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Scalable control of positive systems

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A R T I C L E   I N F O

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Classical control theory does not scale well for large systems such as power networks, traffic networks and chemical reaction networks. However, many such applications in science and engineering can be efficiently modeled using the concept of positive systems and the nonlinear counterpart monotone systems. It is therefore of great interest to see how such models can be used for control.

This paper demonstrates how positive systems can be exploited for analysis and design of large-scale control systems. Methods for synthesis of distributed controllers are developed based on linear Lyapunov functions and storage functions instead of quadratic ones. The main results are extended to frequency domain input–output models using the notion of positively dominated system. Applications to transportation networks and vehicle formations are provided.

1. Introduction

The theory of positive systems and nonnegative matrices has a long history, dating back to the Perron–Frobenius Theorem in 1912. A classic book on the topic is [2]. The theory is used in Leontief economics [11], where the states denote nonnegative quantities of commodities. It also appears in the study of Markov chains [21], where the states denote nonnegative probabilities and in compartment models [9] to model for example populations of species. A nonlinear counterpart is the theory for monotone systems, characterized by the property that a partial ordering of initial states is preserved by the dynamics. Such dynamical systems were studied in a series of papers by Hirsch [13,14].

Positive systems have also been studied in the control literature [23,6,10], and increasingly so during the last decade. Feedback stabilization of positive linear systems was studied in [5]. Stabilizing static output feedback controllers were parameterized using linear programming in [16,15] and extensions to input–output gain were given in [24,4]. Tanaka and Langbort [22] proved that the input–output gain of positive systems can be evaluated using a diagonal quadratic storage function and utilized this for $H_{\infty}$ optimization of decentralized controllers in terms of semi-definite programming.

This paper builds on several contributions by the author [17–19], deriving theory that is applicable to control systems of very large scale. Such systems appear for example in traffic networks, power networks and chemical reaction networks. Classical methods for multi-variable control, such as linear quadratic control and $H_{\infty}$-optimization, do not scale well. The difficulties are partly due to computational complexity and partly to the absence of distributed structure in the resulting controllers. The complexity growth can be traced back to the fact that stability verification of a linear system with $n$ states generally requires a Lyapunov function involving $n^2$ quadratic terms, even if the system matrices are sparse. As was pointed out in [17], the situation improves drastically if we restrict attention to positive systems. Then stability and input–output gain can be verified using a Lyapunov function with only $n$ linear terms. Sparsity can be exploited and even synthesis of distributed controllers can be done with a complexity that grows linearly with the number of nonzero entries in the system matrices. As will be demonstrated in this paper, these observations have far-reaching implications for control engineering:

1. The conditions that enable scalable solutions hold naturally in many important application areas, such as stochastic systems, economics, transportation networks, chemical reactions, power systems and ecology.
2. The essential mathematical property can be extended to frequency domain models, using the concept of “positively dominated” transfer function.
3. The assumption of positive dominance need not hold for the open loop process. Instead, a large-scale control system can often be structured into local control loops that give positive dominance, thus enabling scalable methods for optimization of global input–output gains.

The paper is structured as follows: Section 2 introduces notation. Stability conditions and input–output bounds for positive systems are reviewed in Section 3. Those results are mostly well known, but some aspects of Proposition 3–5 are new. The use in
scalable verification of transportation networks and vehicle formations is introduced in Section 4. Similar ideas are then exploited in Section 5 for synthesis of stabilizing optimal controllers using distributed linear programming. The contributions, Theorems 6 and 7, can be viewed as generalizations of results in [4]. Finally, Section 6 extends the techniques to frequency domain input–output models using the notion of positively dominated transfer function. Further applications to vehicle platoons and transportation networks are given. Summarizing conclusions are given before an appendix with proofs and references.

2. Notation and terminology

Let \( \mathbb{R}_+ \) denote the set of nonnegative real numbers. For \( x \in \mathbb{R}^n \), let \( |x| \in \mathbb{R}^n_+ \) be the element-wise absolute value. This notation should not be confused with the vector norms \( |w|_p = (|w_1|^p + \cdots + |w_m|^p)^{1/p} \) for \( p \in (0, \infty) \) and \( |w|_\infty = \max_i |w_i| \). Given \( M \in \mathbb{C}^{r \times m} \), define the induced matrix norm

\[
\|M\|_{p-\text{ind}} = \sup_{w \in \mathbb{C}^n_+} \frac{|Mw|_p}{|w|_p}.
\]

The spectral norm \( \|M\|_{2-\text{ind}} \) will often just be denoted \( \|M\|_\infty \). Let \( \delta(t) \) be the Heaviside step function and \( \xi(t) \) the Dirac delta function. Then the transfer matrix \( G(s) = C(sI - A)^{-1}B + D \) has the impulse response \( g(t) = Ce^{At}B\delta(t) + De^{At} \). With \( w \in \mathbb{L}^p_+([0, \infty)) \), let \( g \ast w \in \mathbb{L}^p_+([0, \infty)) \) be the convolution of \( g \) and \( w \) and define the induced norms

\[
\|g\|_{p-\text{ind}} = \sup_{w \in \mathbb{L}^p_+([0, \infty))} \frac{\|g \ast w\|_p}{\|w\|_p},
\]

where \( \|w\|_p = \left( \sum_{k=1}^\infty |w_k|^p dt \right)^{1/p} \) for \( p \in (0, \infty) \) and \( \|w\|_\infty = \sup_j \max_i |w_{ij}| \). Let \( \|G\|_{p-\text{ind}} = \|g\|_{p-\text{ind}} \), where \( g \) is the impulse response of \( G \). The norm \( \|G\|_{2-\text{ind}} \) is often called the \( H_\infty \) norm, denoted \( \|G\|_\infty \). It is well known that \( \|G\|_\infty = \sup_{\omega \neq 0} \|G(i\omega)\| \).

The notation 1 means a column vector with all entries equal to the one. The inequality \( X > 0 \) (\( X \geq 0 \)) means that all elements of the matrix (or vector) \( X \) are positive (nonnegative). For a symmetric matrix \( X \), the inequality \( X > 0 \) means that the matrix is positive definite. The matrix \( A \in \mathbb{R}^{n \times n} \) is said to be Hurwitz if all eigenvalues have negative real part. It is Schur if all eigenvalues are strictly inside the unit circle. Finally, the matrix is said to be Metzler if all off-diagonal elements are nonnegative. The notation \( \mathbb{R}^{+m} \) represents the set of rational functions with real coefficients and without poles in the closed right half plane. The set of \( n \times m \) matrices with elements in \( \mathbb{R}^{+m} \) is denoted \( \mathbb{R}^{+m} \). The state space model

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

is said to be an internally positive system if \( A \) is Metzler and \( B, C, D \geq 0 \). It is called an externally positive system if the impulse response \( Ce^{At}B\delta(t) + De^{At}(t) \) is nonnegative. A transfer matrix \( G \) is called positively dominated if every matrix entry satisfies \( |G_{ij}(i\omega)| \leq |G_{ij}(0)| \) for all \( \omega \in \mathbb{R} \).

The term positive system will not be given a precise definition. However, internally positive systems, externally positive systems and positively dominated systems will all be viewed as instances of positive systems. The essential property is that there exists a positive cone in the signal space which is left invariant by the input–output map.

3. Preliminaries

This section introduces some preliminary results on positive systems. Propositions 1 and 2 are well known in the literature since before. References are given in the appendix. Propositions 3–5 are partly known from [4], partly new. Full proofs are given in the same appendix.

**Proposition 1.** Given a Metzler matrix \( A \in \mathbb{R}^{n \times n} \), the following statements are equivalent:

(1.1) The matrix \( A \) is Hurwitz.

(1.2) There exists \( \xi \in \mathbb{R}_+^n \) such that \( A\xi > 0 \).

(1.3) There exists \( z \in \mathbb{R}^n \) such that \( Az > 0 \).

(1.4) There exists a diagonal matrix \( P \) such that \( A^TP + PA < 0 \).

Remark 1. Each of the conditions (1.2), (1.3) and (1.4) corresponds to a Lyapunov function of a specific form. If \( A\xi < 0 \), then \( V(x) = \max_i (x_i/\xi_i) \) is a Lyapunov function with rectangular level curves. If \( z^TA < 0 \), then \( V(x) = z^T|x| \) is a Lyapunov function which is linear in the positive orthant. Finally if \( A^TP + PA < 0 \) and \( P > 0 \), then \( V(x) = x^TPx \) is a quadratic Lyapunov function for the system \( \dot{x} = Ax \). See Fig. 1.

A discrete time counterpart will also be used:

**Proposition 2.** For \( B \in \mathbb{R}_+^{n \times m} \), the following statements are equivalent:

(2.1) The matrix \( B \) is Schur stable.

(2.2) There is a \( \xi \in \mathbb{R}_+^m \) such that \( B\xi > 0 \).

(2.3) There exists \( z \in \mathbb{R}^n \) such that \( Bz > 0 \).

(2.4) There is a diagonal \( P > 0 \) such that \( B^TPB < P \).

(2.5) \( (I - B)^{-1} \) exists and has nonnegative entries.

Moreover, if \( \xi = (\xi_1, \ldots, \xi_n) \) and \( z = (z_1, \ldots, z_n) \) satisfy the conditions of (2.2) and (2.3) respectively, then \( P = \text{diag}(z_1/\xi_1, \ldots, z_n/\xi_n) \) satisfies the conditions of (2.4).

To quantify control performance, it is useful to also discuss input–output gains. A remarkable feature of positive systems is that the input–output gain is determined by the static behaviour [17]:

Fig. 1. Level curves of Lyapunov functions corresponding to the conditions (1.2), (1.3) and (1.4) in Proposition 1. See Remark 1.
Proposition 3. Given matrices $A, B, C, D$ and $g(t) = Ce^{At}Bh(t) + D\delta(t)$, suppose that $Ce^{At}B > 0$ for $t \geq 0$ and $D \geq 0$, while $A$ is Hurwitz. Then $\|g\|_{p,\text{ind}} = \|G(0)\|_{p,\text{ind}}$ for $p = 1, p = 2$ and $p = \infty$. In particular, if $g$ is scalar, then $\|g\|_{p,\text{ind}} = G(0)$ for all $p \in [1, \infty]$.

State-space conditions for input–output gains can now be stated in parallel to the previous stability conditions:

Proposition 4. Let $g(t) = Ce^{At}Bh(t) + D\delta(t)$ where $A \in \mathbb{R}^{n \times n}$ is Metzler and $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$. Then the following statements are equivalent:

1. $\|g\|_{1,\text{ind}} < \gamma$. 
2. There exists $\xi \in \mathbb{R}^m$ such that $\xi > 0$ and
   $$
   \begin{bmatrix}
   A & B \\
   C & D
   \end{bmatrix}
   \begin{bmatrix}
   \xi \\
   1
   \end{bmatrix}
   <
   \gamma
   \begin{bmatrix}
   1 \\
   1
   \end{bmatrix}.
   $$

Moreover, if $\xi$ satisfies (1), then $|x(t)| \leq \xi$ for all solutions to the equation $\dot{x} = Ax + Bw$ with $|x(0)| \leq \xi$ and $\|w\|_\infty \leq 1$.

Proposition 5. Suppose that $g(t) = Ce^{At}Bh(t) + D\delta(t)$ where $A \in \mathbb{R}^{n \times n}$ is Metzler and $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$. Then the following statements are equivalent:

1. $\|g\|_{1,\text{ind}} < \gamma$. 
2. There exists $p > 0$ and
   $$
   \begin{bmatrix}
   A & B \\
   C & D
   \end{bmatrix}
   \begin{bmatrix}
   p \\
   1
   \end{bmatrix}
   <
   \gamma
   \begin{bmatrix}
   1 \\
   1
   \end{bmatrix}.
   $$

Moreover, if $p$ satisfies (2), then all solutions to the equation $\dot{x} = Ax + Bw$ satisfy

$$
\|x(t)\| \leq \int_0^t \|Cw(t) + D\delta(t)\| \, dt \leq \left( \|C\| + p \right) \|x(0)\| + \gamma \int_0^t \|w(t)\| \, dt.
$$

4. Application to scalable verification

Many large scale systems are described by matrices with a sparsity pattern corresponding to a graph. A distributed performance test can then be constructed by assigning each row of the vector inequalities in (1.2), (1.3), (2.2), (2.3) and (3.2) to a corresponding graph node. Then the only local model parameters need to be available to verify the inequalities. The full model is not needed anywhere. This is of course of fundamental importance for manageable computations in many situations.

Also finding a solution to the inequalities, formally a linear programming problem, can be done with distributed methods, where each node in the graph runs a local algorithm involving only local variables and information exchange only with its neighbors. For example, given a stable Metzler matrix $A$, consider the problem to find a stability certificate $\xi > 0$ satisfying $AZ < 0$. This can be done in a distributed way by simulating the system using Euler’s method until the state is close to a dominating eigenvector of $A$. Then it must satisfy the conditions on $\xi$.

Example 1 (Linear transportation network). Consider a dynamical system interconnected according to the graph illustrated in Fig. 2

![Fig. 2. A graph of an interconnected system. In Example 1 the interpretation is a transportation network and each arrow indicates a transportation link. In Example 2 the interpretation is instead a vehicle formation and each arrow indicates the use of a distance measurement.](image)

The model could for example be used to describe a transportation network connecting four buffers. The states $x_1, x_2, x_3, x_4$ represent the contents of the buffers and the gain $\ell_i$ determines the rate of transfer from buffer $j$ to buffer $i$. Such transfer between buffers is necessary to stabilize the system.

Notice that the dynamics has the form $\dot{x} = Ax$ where $A$ is a Metzler matrix provided that every $\ell_{ij}$ is nonnegative. Hence, by Proposition 1, stability is equivalent to existence of numbers $\xi_1, \ldots, \xi_4 > 0$ such that

$$
\begin{bmatrix}
-1 - \ell_{31} & \ell_{12} & 0 & 0 \\
0 & -\ell_{12} - \ell_{32} & \ell_{23} & 0 \\
\ell_{31} & \ell_{12} & -\ell_{23} - \ell_{43} & \ell_{34} \\
0 & 0 & \ell_{43} & -4 - \ell_{43}
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix}
<
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
$$

Given these numbers, stability can be verified by a distributed test where the first buffer verifies the first inequality, the second buffer verifies the second and so on. In particular, the relevant test for each buffer only involves parameter values at the local node and the neighboring nodes.

Example 2 (Vehicle formation (or distributed Kalman filter)). A formation of four vehicles can be described by the following model:

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=\begin{bmatrix}
-1 - \ell_{31} & \ell_{12} & 0 & 0 \\
0 & -\ell_{12} - \ell_{32} & \ell_{23} & 0 \\
\ell_{31} & \ell_{12} & -\ell_{23} - \ell_{43} & \ell_{34} \\
0 & 0 & \ell_{43} & -4 - \ell_{43}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
+ \begin{bmatrix}
\ell_{13} & \ell_{14} & 0 & 0 \\
0 & \ell_{23} & \ell_{24} & 0 \\
0 & 0 & \ell_{34} & 0 \\
-\ell_{13} & -\ell_{14} & 0 & \ell_{43}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}.
$$

Each state is the position of a vehicle. The parameters $\ell_{ij}$ represent position adjustments based on distance measurements between the vehicles. The terms $-x_1$ and $-4x_4$ reflect that the first and fourth vehicle can maintain stable positions on their own, but the second and third vehicle rely on the distance measurements for stabilization. Again, stability can be verified by a distributed test where the first vehicle verifies the first inequality, the second vehicle verifies the second inequality and so on.

Notice that the vehicle formation example is dual to the transportation network example in the sense that the system matrices of (6) and (4) are transpose of each other.

5. Distributed control synthesis by linear programming

Equipped with scalable analysis methods for stability and input–output gains, we are now ready to consider synthesis of controllers by distributed optimization. To do this, we will first revisit the transportation network.
**Example 3.** Consider again (4), with the objective to find stabilizing values of the parameters \(\varepsilon \) in the interval \([0, 1]\). This means that (5) needs to be solved for both \(\varepsilon \) and \(\xi \) simultaneously. At first sight, this looks like a difficult problem due to multiplications between the two categories of parameters. However, a closer look suggests the introduction of new variables \(\mu \) instead of \(\varepsilon \). Then the problem reduces to linear programming:

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & \xi_1 \\
0 & 0 & 0 & 0 & \xi_2 \\
0 & 0 & 0 & -4 & \xi_3 \\
0 & 0 & 0 & 0 & \xi_4
\end{bmatrix}
+ \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
1 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\mu_{13} \\
\mu_{12} \\
\mu_{23} \\
\mu_{34} \\
\mu_{43}
\end{bmatrix}
< 0
\]

In this specific case, there are many solutions, due to the fact that there are many stabilizing controllers. Hence it is of interest to go beyond stability and study also performance in terms of input–output gain.

The following theorem parameterizes all feedback gains such that the closed-loop system remains internally positive and satisfies a given bound on the input–output gain (See Fig. 3).

**Theorem 6.** Let \(\mathcal{D} \) be the set of \(m \times m\) diagonal matrices with entries in \([0, 1]\). Consider matrices \(A, B, C, D, E, F, G, H\) such that \(A + ELF\) is Metzler and \(C + GLF, G + GLH \geq 0\), \(B + ELH \geq 0\), \(D + GLH \geq 0\) for all \(L \in \mathcal{D}\). Let \(g_L(t)\) be the impulse response of the transfer matrix

\[
(C + GLF)[sI - (A + ELF)]^{-1}(B + ELH) + D + GLH
\]

If \(F, H \geq 0\), then the following two conditions are equivalent:

\[
(6.1) \text{There exists } L \in \mathcal{D} \text{ such that } A + ELF \in \mathbb{R}^{m \times m} \text{ Hurwitz and } \|g_L(t)\|_{\infty \rightarrow \infty} < \gamma.
\]

\[
(6.2) \text{There exist } \xi \in \mathbb{R}^m_+ \text{, } \mu \in \mathbb{R}^m_+ \text{ such that } A\xi + B\mu < 0, C\xi + D\mu < \gamma 1, F\xi + H1 \geq \mu.
\]

Moreover, if \(\xi, \mu\) satisfy (6.2), then (6.1) holds for every \(L\) with \(\mu = L(F\xi + H1)\).

Remark 2. If the diagonal elements of \(\mathcal{D}\) are restricted to \(\mathbb{R}_+\) instead of \([0, 1]\), then the condition \(F\xi + H1 \geq \mu\) is replaced by \(F\xi + H1 \geq 0\).

**Remark 3.** It is interesting to compare our results with the analysis and synthesis methods proposed in [22] and [4]. The main difference is our “static output feedback” expression \(A + ELF\), which is significantly more general than the “state feedback” expression \(A + BL\) used in those references. In particular, the matrix \(F\) creates coupling between the columns of \(A + ELF\) that cannot be enforced in \(A + BL\). On the other hand, the second parametrization makes it possible to enforce the Metzler property of the closed loop system as a constraint in the synthesis procedure, rather than being verified a priori for all \(L \in \mathcal{D}\).

**Remark 4.** A natural question related to Theorem 6 is how restrictive it is to require internal positivity of the closed loop system. A complete answer to this question is very difficult to get, simply because no general methods for synthesis of optimal decentralized controllers are available. However, our experience from examples indicate that there is a large and important class of systems for which the assumption of closed loop positivity makes no difference whatsoever in terms of achievable performance. This will a topic of further research in the future.

**Proof.** Suppose (6.1) holds. Then, according to Proposition 4, there exists \(\xi \in \mathbb{R}^m_+\) such that

\[
\begin{bmatrix}
A + ELF & B + ELH \\
C + GLF & D + GLH
\end{bmatrix}
[\xi \mu] < 0 \begin{bmatrix}1 \gamma \end{bmatrix}.
\]

Setting \(\mu = LF\xi + LH1\) gives (6.2). Conversely, suppose that (6.2) holds. Choose \(L \in \mathcal{D}\) to get \(\mu = LF\xi + LH1\). Then (7) holds and (6.1) follows by Proposition 4.

**Theorem 6** was inspired by the transportation network in Example 3, where non-negativity of \(F\) is natural assumption. However, this condition would fail in a vehicle formation problem, where control is based on distance measurements. For such problems, the following dual formulation is useful:

\[
\begin{bmatrix}
A + ELF & B + ELH \\
C + GLF & D + GLH
\end{bmatrix}
[\xi \gamma] < 0 \begin{bmatrix}1 \mu \end{bmatrix}.
\]

Setting \(\mu = LF\xi + LH1\) gives (6.2). Conversely, suppose that (6.2) holds. Choose \(L \in \mathcal{D}\) to get \(\mu = LF\xi + LH1\). Then (7) holds and (6.1) follows by Proposition 4.

**Theorem 7.** Let \(\mathcal{D} \) be the set of \(m \times m\) diagonal matrices with entries in \([0, 1]\). Consider matrices \(A, B, C, D, E, F, G, H\) such that \(A + ELF\) is Metzler and \(C + GLF, G + GLH \geq 0\), \(B + ELH \geq 0\), \(D + GLH \geq 0\) for all \(L \in \mathcal{D}\). Let \(g_L(t)\) be the impulse response of the transfer matrix

\[
(C + GLF)[sI - (A + ELF)]^{-1}(B + ELH) + D + GLH
\]

If \(E, G \geq 0\), then the following two conditions are equivalent:

\[
(7.1) \text{There exists } L \in \mathcal{D} \text{ such that } A + ELF \in \mathbb{R}^{m \times m} \text{ Hurwitz and } \|g_L(t)\|_{\infty \rightarrow \infty} < \gamma.
\]

\[
(7.2) \text{There exist } p \in \mathbb{R}^m_+ \text{, } q \in \mathbb{R}^m_+ \text{ such that } A^2p + C^21 + F^2q < 0, B^2p + D^21 + H^2q < \gamma 1, E^2p + G^21 \geq q.
\]

If \(p, q\) satisfy (7.2), then (7.1) holds for every \(L\) such that \(q = LE^2p + G^21\).

**Proof.** The proof is analogous to the proof of Theorem 6.

**Example 4** (Disturbance rejection in vehicle formation). Consider again the vehicle formation model

\[
\begin{aligned}
x_1 &= -x_1 + \ell_{13}(x_3 - x_1) + v_1 \\
x_2 &= \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) + v_2 \\
x_3 &= \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + v_3 \\
x_4 &= -4x_4 + \ell_{43}(x_3 - x_4) + v_4
\end{aligned}
\]

now with \(v\) as an external disturbance acting on the vehicles. Our problem is to find feedback gains \(\varepsilon \in [0, 1]\) that stabilize the formation and minimize the influence of the disturbance \(v\). This can be done by applying Theorem 7 with

\[
A = \text{diag}[-1, 0, 0, -4], \quad B = I
\]
to minimize the 1-induced input–output gain from $\nu$ to $x$. When $B=I$, all four vehicles face disturbances of the same size and the optimal $L=\text{diag}(0,1,1,0,1,0)$ illustrated by arrows in the left diagram of Fig. 4 gives $\gamma=4.125$. Apparently, the first vehicle should ignore the distance to the third vehicle, while the third vehicle should ignore the second vehicle and the fourth should ignore the third. The middle diagram of Fig. 4 illustrates a situation where the disturbances on vehicle 1 and 2 are 10 times bigger. Then the minimal value $\gamma=15.562$ is attained with $L=\text{diag}(1,1,1,0,1,0)$, so the first vehicle should use distance measurements to the third. The converse situation in the right diagram gives $\gamma=12.750$ for $L=\text{diag}(0,1,1,0,1,0)$. □

Transportation and formation problems as described above are important application areas where positive systems appear naturally. However, there are also many cases where the open loop system is not positive, but where theory for positive systems is useful for design and verification of the closed loop. To see this, we will in the next section extend the notion of positive system to frequency domain input–output models.

6. Positively dominated systems

In the study of input–output models, one option would be to work with externally positive systems, i.e. linear time-invariant systems with nonnegative impulse response. However, to verify external positivity of a given rational transfer function has proved to be NP-hard. See [3] for the discrete time problem and [1] for continuous time. Instead we will use the notion of positively dominated transfer matrix $G$, meaning that $|G_b(a\omega)| \leq G_a(0)$ for all $a \in \mathbb{R}$. The essential scalar frequency inequality can be tested by semi-definite programming, since $|b(a\omega)/a(\omega)| \leq b(0)/a(0)$ holds for $a \in \mathbb{R}$ if and only if the polynomial $(b(a\omega))^2-a(\omega)^2$ can be written as a sum of squares. Define $\mathbb{D}^{m,n}_{\infty}$ as the set of $m \times n$ matrices with entries that are stable proper positively dominated rational functions. Some properties follow immediately:

**Proposition 8.** Let $G, H \in \mathbb{D}^{m,n}_{\infty}$. Then $GH \in \mathbb{D}^{m,n}_{\infty}$ and $aG+bH \in \mathbb{D}^{m,n}_{\infty}$ when $a,b \in \mathbb{R}_+$. Moreover

$$\sup_{\omega} |G_b(a\omega)|_{p-\text{ind}} = \|G(0)\|_{p-\text{ind}}, \quad p \in [1, \infty]$$

The following property is also fundamental:

**Proposition 9.** Let $G \in \mathbb{D}^{m,n}_{\infty}$. Then $(I-G)^{-1} \in \mathbb{D}^{m,n}_{\infty}$ if and only if $G(0)$ is Schur.

**Proof.** $(I-G)^{-1}$ being positively dominated implies that $|I-G(0)|^{-1}$ exists and is nonnegative, so $G(0)$ must be Schur according to Proposition 2. For every $z \in \mathbb{C}^n$ and $s \in \mathbb{C}$ with $Re\,s \geq 0$ the assumption $G \in \mathbb{D}^{m,n}_{\infty}$ gives

$$|G(s)z| \leq |G(0)|z$$

for $k=1,2,3,\ldots$

Hence $\sum_{k=0}^{\infty} G(s)^k z$ is convergent and bounded above by $\sum_{k=0}^{\infty} G(0)^k z = |I-G(0)|^{-1} z$. The sum of the series solves the equation $|I-G(s)|z = z$, so therefore $\sum_{k=0}^{\infty} G(s)^k z = |I-G(s)^{-1} z$. This proves $(I-G)^{-1}$ is stable and positively dominated and the proof is complete. □

As we will see, many important large scale control systems can be analyzed effectively by exploiting the concept of positive dominance.

**Example 5** (Platoon of vehicles with inertia). Using positive dominance, it is possible to consider more general vehicle models than in [8]. For example, consider the following model of four interacting vehicles, including inertia, damping and dynamic feedback:

$$(s^2+0.15)x_1 = -C_1(s)\dot{x}_1 + v_1$$

$$(s^2+0.15)x_2 = C_2(s)(\lambda_{21}\dot{x}_1 - x_2) + \lambda_{23}(x_1 - x_2) + \lambda_{24}(x_1 - x_2) + v_2$$

$$(s^2+3)x_3 = C_3(s)(\lambda_{32}\dot{x}_1 - x_3) + \lambda_{34}(x_4 - x_3) + v_3$$

$$(s^2+3)x_4 = -C_4(s)x_4 + v_4$$

Here $\lambda_{ik} \in [0,1]$ with $\lambda_{21} + \lambda_{23} = \lambda_{32} + \lambda_{34} = 1$. The interpretation is that the second and third vehicle are controlled based on a weighted average of the distances to neighboring vehicles, while the first and fourth vehicle can measure their own position. All vehicles have the same inertia, but the first and second vehicles have less damping than the others. Using the notation

$$C = \text{diag}(C_1, C_2, C_3, C_4)$$

$$P = \text{diag}(P_1, P_2, P_3, P_4) = \text{diag}\left(\frac{1}{s^2+0.15}, \frac{1}{s^2+0.15}, \frac{1}{s^2+3}, \frac{1}{s^2+3}\right)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_{21} & 0 & 1 - \lambda_{21} & 0 \\ 0 & \lambda_{32} & 0 & 1 - \lambda_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the closed loop system can be written on the form

$$x = P(s)C(s)[A - I]x + P(s)v$$

where the controllers $C(s)$ and the weights $A$ are available for

![Fig. 4. Illustration of optimal gains for disturbance rejection in a vehicle formation. Dotted arrows correspond to distance measurements that are not used in the optimal solution.](image-url)
design. Our approach is to first make \( P_i(s)C_i(s)/[1 + P_i(s)C_i(s)] \) positively dominated by selecting the local controllers \( C_i(s) \). This property can then be exploited in the second stage to optimize the weights \( \Lambda \) in a scalable way. See Fig. 5.

**Theorem 10.** Given diagonal transfer matrices \( L(s), W'(s), W''(s) \) and a nonnegative matrix \( \Lambda \) be with spectral radius smaller than one, consider the interconnected system

\[
x = L(s)[(\Lambda - I)x + W'(s)v] + W''(s)v.
\]

The closed loop transfer function from \((v, w)\) to \( x \) is equal to

\[
G = (I - \Lambda)^{-1}[SW' \quad TW'']
\]

where \( T = L(I + L)^{-1} \) and \( S = (I + L)^{-1} \). Suppose that \( T \) is stable and positively dominated with \( T(0) = I \). Then \( G \) is stable and

\[
\sup_{\omega} \| G(i\omega) \|_{p - \text{ind}} \leq \| (I - \Lambda)^{-1} \Gamma \|_{p - \text{ind}} \tag{10}
\]

where \( \Gamma = \text{diag}\{\gamma_1, \ldots, \gamma_m\} \) and \( \gamma_k = \| [S]_{k1} W'_{k1} T_{1k} W''_{1k} \|_{\infty} \). If also \( SW' \) and \( TW'' \) are positively dominated, then (10) holds with equality.

**Remark 5.** If both controllers \( C \) and weights \( \Lambda \) are available for optimization, then the global optimum can be found by first minimizing each \( \gamma_k \) separately by choosing \( C_k \), then computing the optimal weight matrix \( \Lambda \). Both stages are convex and scalable. Optimization of \( C_k \) is convex in the Youla parametrization since the set of positively dominated \( T_k \) is convex. However, further research is needed on the computational aspects.

**Proof.** The system (9) can be rewritten

\[
(I + L)x = L[Ax + W'w] + W''v
\]

\[
(I - \Lambda)x = [SW' \quad TW''] \begin{bmatrix} v \\ w \end{bmatrix}
\]

\[
x = (I - \Lambda)^{-1} [SW' \quad TW''] \begin{bmatrix} v \\ w \end{bmatrix}
\]

\( T \) is assumed to be positively dominated with \( T(0) = I \), so by Proposition 9, also \((I - \Lambda)^{-1}\) is positively dominated and its elements are bounded above by the elements of \((I - \Lambda)^{-1}\). In particular,

\[
\sup_{\omega} \| G(i\omega) \| = \| (I - \Lambda)^{-1} [S(i\omega)W'(i\omega) \quad T(i\omega)W''(i\omega)] \| \\
\leq \| (I - \Lambda)^{-1} \Gamma \|
\]

for all \( \omega \). Equality follows if also \( SW' \) and \( TW'' \) are positively dominated.

The following proposition can be used for verification of the assumptions.

**Proposition 11.** Suppose that \( L \) is diagonal and has no poles in the right half plane except for one at the origin. Then \((I + L)^{-1}\) is positively dominated if and only if \( \text{Re } L(j\omega) \geq -\frac{1}{2} \) for all \( j \) and \( \omega \in \mathbb{R} \).

**Example 5 continued.** Choosing for example

\[
C_1(s) = C_2(s) = 0.1 + \frac{2s}{s + 2}
\]

\[
C_3(s) = C_4(s) = \frac{4}{9}
\]

makes \( T \) and \( SW' \) positively dominated. Application of Theorem 10 with \( L = PC \) and \( W'' = P \) gives that the \( H_\infty \)-norm of the map from \( v \) to \( x \) is equal to

\[
\| (I - \Lambda)^{-1} \Gamma \|
\]

where \( \Gamma = \text{diag}\{10, 10, 4, 4, 4, 4\} \). The expression \( \| (I - \Lambda)^{-1} \Gamma \| \) is convex in \( \Lambda \) (due to the nonnegative entries), so it is straightforward to compute the optimal weights \( \lambda_{21} = 0.12 \) and \( \lambda_{32} = 0 \). (Notice that another choice of controllers \( C_k \) generally would lead to other optimal weights.)

**Example 6 (Heterogeneous vehicle string with one leader).** A one-dimensional string of \( m \) vehicles can be modeled as follows:

\[
\begin{cases}
x_1 = P_1 C_1(W_1^1)W_1^1 w_1 - x_1 + W'_1 v_1 \\
x_k = P_k C_k(W_k^1 w_k + c_k x_k + (1 - \epsilon)x_{k-1} - x_k) + W'_k v_k
\end{cases}
\]

for \( k = 2, \ldots, m \). Here \( P_1, \ldots, P_m \) are transfer functions from control inputs to vehicle positions, while \( C_1, \ldots, C_m \) are controller transfer functions. Here, only the first vehicle can measure absolute position, while the remaining vehicles rely on distance measurements. More precisely, their control action is based on a weighted average of the distance to the preceding vehicle and the distance to the leading vehicle of the platoon. The weight \( \epsilon \) can be said to represent the level of confidence in information about the distance to the leading vehicle.

Measurement noise is modeled by the \( W'_j w_j \)-terms, so \( W' \) would typically be high frequency dominated, while \( W'' \) is low frequency dominated to get good rejection of static position errors (integral action). With

\[
L = \text{diag}\{P_1 C_1, \ldots, P_m C_m\}
\]

\[
W' = \text{diag}\{W'_1, \ldots, W'_m\}
\]

\[
W'' = \text{diag}\{W''_1, \ldots, W''_m\}
\]

we are in the setting of Theorem 10. To satisfy the constraint \( T(0) = I \), we need to use controllers with integral action (unless there is an integrator in the process). To make \( T \) positively dominated, we also need to keep \( \text{Re } P_i(i\omega) C_i(i\omega) \geq -\frac{1}{2} \) for all \( j \) and \( \omega \in \mathbb{R} \). The remaining constraints can also be dealt with one controller at a time.
A network connecting producers and consumers is sometimes called a network. An example taking the following form:

\[ \begin{align*}
    \Lambda & = \begin{bmatrix}
        0 & \lambda_{12} & 0 & 0 \\
        0 & 0 & \lambda_{23} & 0 \\
        0 & 1 - \lambda_{12} & 0 & 0 \\
        0 & 0 & 1 - \lambda_{23} & 0
    \end{bmatrix} \\
    C & = \text{diag}(C_1, C_2, C_3, C_4) \\
    P & = \text{diag}(P_1, P_2, P_3, P_4) = \text{diag}\left\{ \frac{1}{s^2 + 0.1s}, \frac{1}{s^2 + 0.15s}, \frac{1}{s^2 + s}, \frac{1}{s^2 + s} \right\}
\end{align*} \]

The closed loop system can be written in the following form:

\[ x = P(s)(\Lambda - I)(C(s)x + P(s)w) \]

Introducing \( u = C(s)x \), the equation can also be written as

\[ u = C(s)P(s)(\Lambda - I)u + w \]

which is again the format where Theorem 10 can be applied. Comparison with Example 5 once again illustrates the duality between formation and transportation problems. In formation problems, the row sums of \( \Lambda \) are bounded above by one, but in transportation problems the bound applies to column sums.\(^\dagger\)

7. Conclusions

The results above demonstrate that the monotonicity properties of positive systems and positively dominated systems bring remarkable benefits to control theory. Most important is the opportunity for scalable verification and synthesis of distributed control systems with optimal input–output gains. In particular, linear programming solutions come with certificates that enable distributed and scalable verification of global optimality, without access to a global model anywhere. Many important problems remain open for future research. Here are some examples:

- Further analyze consequences of the closed loop positivity assumption. For what systems is it non-restrictive?\(^\dagger\)
- Derive efficient methods for optimization of local controllers with positively dominated closed loop dynamics.
- Extend the scalable methods for verification and synthesis to monotone nonlinear systems.

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Appendix A

A.1. References and proofs

Good general references for basics results on positive systems are [2,7]. Below we give more detailed references and in some cases full proofs for some basic facts.

References for Proposition 1. The equivalence between (1.1), (1.2), (1.4) and (1.5) is the equivalence between the statements \( G_{20}, I_{27}, H_{24} \) and \( N_{38} \) in [2, Theorem 6.2.3]. The equivalence between (1.1) and (1.3) is obtained by applying the equivalence between (1.1) and (1.2) to the transpose of \( A \). Moreover, if \( \xi = (\xi_1, \ldots, \xi_n) \) and \( z = (z_1, \ldots, z_n) \) satisfy the conditions of (1.2) and (1.3) respectively, then \( P = \text{diag}(z_1/\xi_1, \ldots, z_n/\xi_n) \) gives \( A^TP + PA\xi = A^Tz + PA\xi < 0 \) so the symmetric matrix \( A^TP + PA \) is Hurwitz and (1.4) follows.\(^\dagger\)

References for Proposition 2. The equivalence between (2.1) and (2.5) is proved by [2, Lemma 6.2.1]. Setting \( A = B - I \) gives the equivalence between (2.2), (2.3) and (2.5) from the equivalence between (1.2), (1.3) and (1.5).
Suppose $\xi = (\xi_1, \ldots, \xi_n)$ and $z = (z_1, \ldots, z_n)$ satisfy the conditions of (2.2) and (2.3) respectively. Set $P = \text{diag}(p_1/\xi_1, \ldots, p_n/\xi_n)$. Then $B^TP\xi < B^TPz < B^TZ < P\xi$ so $B^TPB - P$ is Hurwitz and (2.4) follows. Finally, (2.4) shows that $x^{\text{TP}x}$ is a positive definite Lyapunov function for the system $x^+ = Bx$, so (2.1) follows from (2.4).\[\Box\]

The cases $p = 1$ and $p = \infty$ of Proposition 3 previously appeared in [4]. The same is true for the first part of Propositions 4 and 5. However, for convenience of the reader, we here give full proofs of all three propositions.

**Proof of Proposition 3.** It is well known that $\|g\|_{2-\text{ind}} = \max_{H \in \mathcal{M}} \|G(\mathbf{i}0)\|_{\infty-\text{ind}}$ for general linear time-invariant systems. When $g(t) \geq 0$, the maximum must be attained at $\omega = 0$ since

$$
\|G(\mathbf{i}0)w\| \leq \int_0^\infty \|g(t)e^{-\omega t}\| dt \cdot \|w\| = \int_0^\infty g(t) dt \cdot \|w\| = G(\mathbf{0}) \|w\|
$$

for every $w \in \mathbb{C}^m$. This completes the proof for $p = 2$. For $p = 1$, the fact follows from the calculations

$$
\|y\|_1 = \sum_k \sum_{\tau \geq 0} \sum_{\tau} g_{kl}(t-\tau) |w_k(\tau)| dt \\
\leq \sum_k \sum_{\tau \geq 0} \sum_{\tau} g_{kl}(t-\tau) |w_k(\tau)| dt \\
= \sum_k \sum_{\tau} \left( \int_0^\infty g_{kl}(t) dt \right) |w_k(\tau)| dt \\
= \sum_k \sum_{\tau} \int_0^\infty g_{kl}(t) dt \left\|w_k(\tau)\right\|_1 \\
\leq \max_k \left( \sum_k \sum_{\tau} \int_0^\infty g_{kl}(t) dt \left\|w_k(\tau)\right\|_1 \\
= \|G(\mathbf{0})\|_{1-\text{ind}} \cdot \|w\|_1
$$

with equality when $\|G(\mathbf{0})\|_{1-\text{ind}} \cdot \|w\|_1 = \|G(\mathbf{0})w\|_1$. Similarly, for $p = \infty$,

$$
\|y\|_\infty = \max_k \left( \sum_k \int_0^\infty g_{kl}(t) dt \right) \left\|w_k(\tau)\right\|_\infty \\
\leq \max_k \left( \sum_k \int_0^\infty g_{kl}(t) dt \right) \left\|w_k(\tau)\right\|_\infty \\
= \max_k \left( \sum_k \|G(\mathbf{0})\|_{\infty-\text{ind}} \cdot \|w\|_\infty
$$

with equality when $w_k(\tau)$ has the same value for all $k$ and $t$. Hence the desired equality

$$
\|g\|_{p-\text{ind}} = \|G(\mathbf{0})\|_{p-\text{ind}}
$$

has been proved for $p = 1$, $p = 2$ and $p = \infty$. In particular, if $g$ is scalar, then for these values of $p$

$$
\|g\|_{p-\text{ind}} = G(\mathbf{0}). \tag{11}
$$

The Riesz–Thorin convexity theorem [8, Theorem 7.1.12] shows that $\|g\|_{p-\text{ind}}$ is a convex function of $p$ for $1 \leq p \leq \infty$, so (11) must hold for all $p \in [1, \infty]$.\[\Box\]

**Proof of Proposition 4.** Assuming that $M$ has nonnegative entries, it is straightforward to verify [7, Lemma 8.1.21] that

$$
\|M\|_{1-\text{ind}} < \gamma \quad \text{if and only if} \quad M^T1 < \gamma 1
$$

$$
\|M\|_{\infty-\text{ind}} < \gamma \quad \text{if and only if} \quad M1 < \gamma 1
$$

Assume that (4.2) holds. Then $A$ is Hurwitz by Proposition 1. Furthermore, $e^{\omega t} \geq 0$ and the assumptions of Theorem 3 hold, so $\|g\|_{\infty-\text{ind}} < \gamma$ can equivalently be written $\|D - CA^{-1}B\|_{1-\text{ind}} < \gamma$ or

$$
(D - CA^{-1}B)1 < \gamma 1. \tag{12}
$$

Multiplying the inequality $A\xi + B1 < 0$ by the non-positive matrix $CA^{-1}$ from the left gives $C\xi + CA^{-1}B1 \geq 0$. Subtracting this from the inequality $C\xi + D1 < \gamma 1$ gives (12), so (4.1) follows.

Conversely, suppose that (4.1) and therefore (12) holds. By Proposition 1 there exists $x \geq 0$ such that $Ax \leq 0$. Define $\xi = x - A^{-1}B1$. Then $\xi \succeq x > 0$. Moreover $A\xi + B1 = Ax < 0$.

If $x$ is sufficiently small, we also get $C\xi + D1 < \gamma 1$ so (4.2) follows. To prove the last statement, suppose that $\xi$ satisfies (1) and define $x$, $y$ and $z$ by

$$
y = Ay + u \quad y(0) = -\xi \tag{13}
y = Ax + Bw \quad |x(0)| \leq \xi \tag{14}
z = Az + v \quad z(0) = \xi \tag{15}
$$

where $\|w\|_\infty \leq 1$, $u = A\xi$ and $v = -A\xi$. Then the solutions of (13) and (15) are constantly equal to $-\xi$ and $\xi$ respectively. Moreover, the inequalities

$$
u \leq Bw \leq v$$

follow from (1). Together with the assumption that $A$ is Metzler, gives that $y(t) \leq x(t) \leq x(t)$ for all $t$. This completes the proof.\[\Box\]

**Proof.** Proof of Proposition 5: Assume that (5.2) holds. Then $A$ is Hurwitz by Proposition 1. Furthermore, $e^{\omega t} \geq 0$ and the assumptions of Theorem 3 hold, so the inequality $\|g\|_{1-\text{ind}} < \gamma$ can equivalently be written $\|D - CA^{-1}B\|_{1-\text{ind}} < \gamma$ or

$$
(D - CA^{-1}B)1 < \gamma 1. \tag{16}
$$

Consider any solutions to

$$x = Ax + Bw \quad y = Ay + B|w| \quad y(0) = |x(0)|.\tag{17}
$$

The matrix $A$ is Metzler, so $|x(t)| \leq y(t)$ for all $t \geq 0$. Multiplying the transpose of (2) by $(y, w)$ from the right gives

$$p^Ty + 1^T(Cy + D)w \leq \gamma 1^T|w|.\tag{18}
$$

Integrating over $t$ gives

$$p^Ty(t) + \int_0^t 1^T(Cy + D)w| \ dt \leq p^Ty(0) + \gamma \int_0^t 1^T|w| \ dt \tag{19}
$$

and using that $|x(t)| \leq y(t)$ one gets (3). Then (5.1) follows as $t \to \infty$. Conversely, suppose that (5.1) holds. By Proposition 1 there exists $z \geq 0$ such that $z^TA < 0$. Define $p = z - A^{-1}C1$. Then $p \succeq z > 0$. Moreover $A^Tp^T1 = A^Tz < 0$.

If $z$ is sufficiently small, we also get $B^Tp + D^T1 < \gamma 1$ so (5.2) follows and the proof is complete.\[\Box\]
References


