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Spatially Coupled Protograph-Based LDPC Codes for Incremental Redundancy

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***Abstract*—We investigate a family of protograph based rate-compatible LDPC convolutional codes. The code family shows improved thresholds close to the Shannon limit compared to their uncoupled versions for the binary erasure channel as well as the AWGN channel. In fact, the gap to Shannon limit is almost uniform for all members of the code family ensuring good performance for all subsequent incremental redundancy transmissions. Compared to similar code families based on regular LDPC codes [1] the complexity of our approach grows slower for the considered rates.**

I. Introduction

The use of incremental redundancy (IR) in modern wireless communication systems is very popular. The varying nature of the wireless communication channel makes an adaptive procedure for forward error correction necessary to support reliable communication. First schemes of that kind where introduced in [2]. To obtain this adaptivity the scheme normally works in the following way. An initial codeword is transmitted and in case of erroneous reception and a failed decoding attempt, the transmitter is requested to send additional redundant information. For obtaining this redundant information several ways were proposed. One approach is to use puncturing. Starting with a mother code of very low code rate, specific bits of a codeword are punctured to obtain a codeword of higher code rate. A rate-adaptive family of codes is then produced by applying different puncturing patterns to the mother code. The problem of puncturing is that the highly optimized mother code is always the lowest rate code while the initial transmission is with high code rate. The optimization in [3] showed an increasing gap to capacity with higher rate resulting in an initial transmission with worse performance. To overcome this issue extension is the opposite approach to puncturing [4]. Beginning with a high-rate code subsequent codewords are obtained by extending the parity matrix. The authors in [5] considered the design of a family of rate-compatible irregular LDPC codes. These codes outperform the previously mentioned punctured counterparts as they show a uniform gap to capacity over a wide range of rates. However with the use of irregular codes, the degree distributions have to be optimized for every rate complicating both design and implementation. The above mentioned solutions fall into the category of finite discrete rate incremental redundancy code designs. The introduction of LT codes [6] and Raptor Codes [7] formed a new coding paradigm of rateless codes which can virtually generate any desired rate. Similarities between rateless and finite discrete rate code designs were already pointed out in [8]. An example of a finite discrete rate code family design that is motivated by the code structure of Raptor codes was recently introduced in [9] and can be referred to as a protograph based rate-compatible LDPC code family that has a raptor-like code structure.

Investigations on the interconnection of LDPC block codes revealed a threshold improvement [10] which is only due to the coupled structure of the resulting graph. The term *spatial coupling* was introduced in [11] together with an analytical treatment of the threshold saturation effect. While this effect was investigated in conjunction with channel coding the same behaviour can be observed in other research areas related to graphical models.

The coupling of an incremental redundancy code family based on the extension of regular LDPC codes was investigated in [1]. For this case the codes are proven to achieve capacity of the binary erasure channel (BEC) [11]. Coupling of rateless codes was investigated in [12], where the authors coupled a LT code ensemble and conjectured to universally achieve capacity on different channels.

In this paper we present a family of protograph-based rate-compatible LDPC convolutional code ensembles for use in incremental redundancy applications and give an analysis of the thresholds and complexity of the code ensembles in comparison to the regular convolutional codes. The considered ensembles are spatially-coupled versions of the code family in [9]. The paper is organized as follows. In Section II the original raptor-like LDPC codes are introduced and a short introduction to spatial coupling is given. Then the family of protograph-based rate-compatible LDPC convolutional codes is described and the procedure to obtain such an ensemble is explained. A detailed threshold analysis is then presented in Section IV, followed by an assessment of the decoding complexity. Finally, the paper is concluded with an outlook.

II. Preliminaries

A. Raptor-like LDPC Codes for Incremental Redundancy

In [9] a family or rate-compatible LDPC codes with a raptor-like graph structure was introduced. The term "raptor-like" is derived from the similar graphical structure that a conventional raptor code shows. A raptor code is formed by a specifically parametrized LT code [6] that is precoded by a properly chosen LDPC code in order to improve the error floor behaviour. For a conventional raptor code the connections of the LT code part are chosen "on the fly" in the encoding process resulting in a random definition of the code during encoding. The following code families differ from this approach as their description is predetermined. To obtain a raptor-like code based on protographs as in [9] first a suitable precode $\mathcal{C}_1$ is chosen, defined by the protograph base

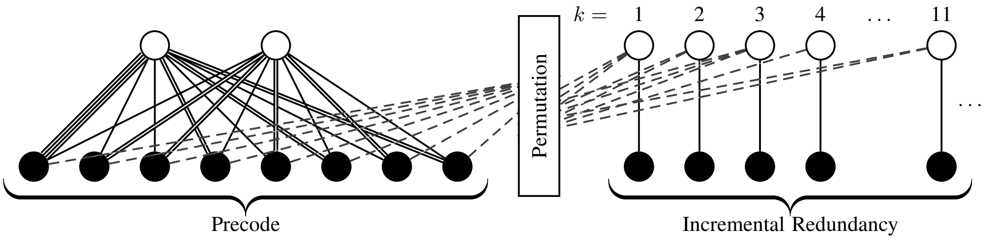

Fig. 1. Exemplary protograph of a raptor-like LDPC code [9]

matrix $\mathbf{B}^{(1)}$ with N_1 variable nodes and M_1 check nodes. To produce a rate-compatible family of codes $\{\mathcal{C}_k\}_1^K$ for K incremental redundancy steps the following procedure is used. For subsequent incremental redundancy transmissions a new variable node and a new check node are appended to the precode and connected to each other with an edge. Then the connections from the newly generated check node to the variable nodes of the precode are chosen such that the threshold is optimized, resulting in a code $\mathcal{C}_k$ for incremental redundancy step k. The procedure obtains a code family with K members and assigned protographs as in Fig. 1, where for incremental redundancy step k the protograph consists of the precode and the incremental redundancy nodes up to index k. For the rest of the paper we consider the exemplary code family from [9] with $K = 11$ code members but the concept can be extended to code families with an arbitrary number of members K.

The graph representation also relates to a description via extended base matrices. Assume the rate-compatible code family $\{\mathcal{C}_k\}_1^K$ with associated base matrices $\mathbf{B}^{(k)}$ for $k \in \{1 \dots K\}$. Note that $\mathbf{B}^{(1)}$ is the base matrix of the precode. Then each $\mathbf{B}^{(k)}$ is given by

$$\mathbf{B}^{(k)} = \begin{bmatrix} \mathbf{B}^{(k-1)} & \mathbf{0}_{M_1+k-1\times 1} \\ \mathbf{B}^{(e,k)} \quad \mathbf{0}_{1\times k-2} & 1 \end{bmatrix};\ k = 2, \dots K \tag{1}$$

where $\mathbf{B}^{(e,k)}$ is representing the optimized connections between newly generated check nodes and the variable nodes of the precode and $\mathbf{0}_{m\times n}$ denotes an all-zero matrix of size $m \times n$. The base matrix for step $k-1$ is part of the base matrix at step k, a necessary condition for rate-compatibility. The recursive construction of base matrices results in a nested structure as depicted in Fig. 2. The codes presented in [9] exhibit very

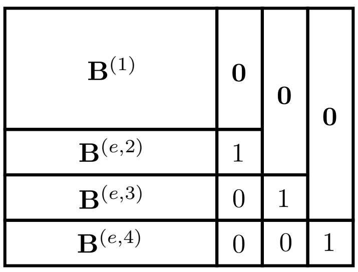

Fig. 2. Extension structure of the base matrix

good thresholds with small gap to the Shannon limit for the AWGN channel and a similar behavior can be found for the BEC. Simulations in [9] show the practical applicability of such code families.

B. Spatial Coupling

The transmission of a sequence of codewords $\mathbf{v}_t$ with $t = 1 \dots L$ using an LDPC block code is considered in the sequel. The code structure can be defined by a protograph with M check- and N variable nodes and an associated base matrix $\mathbf{B}$. The fundamental difference between a LDPC block code and its convolutional version is, that in the latter case codewords of different time instants are interconnected. The codewords are *coupled* over several time instants t. The memory m_s of the coupled ensemble defines the maximal distance between two coupled blocks. To obtain an LDPC convolutional code from a given chain of LDPC block codes a procedure called *edge spreading* was introduced in [13]. Edges from variable nodes at index t are spread among check nodes in the range $t, t+1, \dots, t+m_s$. The resulting protograph based terminated LDPC convolutional code ensemble can be described by its convolutional base matrix

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} \mathbf{B}_0 & & \\ \vdots & \ddots & \\ \mathbf{B}_{m_s} & & \mathbf{B}_0 \\ & \ddots & \vdots \\ & & \mathbf{B}_{m_s} \end{bmatrix}_{(L+m_s)M\times LN} \tag{2}$$

Uncoupled LDPC block codes are a special case with $\mathbf{B}_0 = \mathbf{B}$ and $m_s = 0$. The edge spreading procedure can be applied to any given protograph of regular or irregular kind with given base matrix. To preserve the original structure of the computation tree together with maintaining the check- and variable node degrees, the condition

$$\mathbf{B} = \sum_{i=0}^{m_s} \mathbf{B}_i \tag{3}$$

must be satisfied. The termination of the chain after L instants introduces a rate loss at the boundaries which vanishes if $L \to \infty$.

III. Coupled Protograph-Based Rate-Compatible LDPC Codes

Now we introduce a family of spatially coupled protograph-based rate-compatible LDPC codes. Spatial coupling is applied to decrease the gap to the Shannon limit for the original raptor-like LDPC codes. The obtained protograph code is a rate-compatible LDPC convolutional code family with raptor-like graph structure. To generate such a convolutional protograph the edges belonging to an original protograph $\mathbf{B}^{(k)}$ are

spread over several time instants and the resulting structure is concatenated L times to generate a chain of spatially connected protographs. A convolutional structure of this kind is figuratively illustrated in Fig. 3.

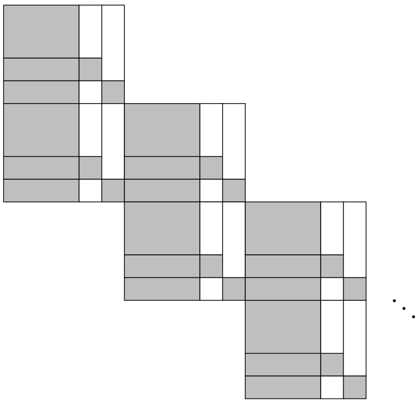

Fig. 3. Banded structure of a rate-compatible LDPC convolutional code

Here white rectangles denote all-zero sub-matrices and shaded rectangles denote non-zero sub-matrices. The chain is terminated after L time instants to form a block code with a convolutional structure and LN_k variable nodes.

To generate a rate-compatible convolutional LDPC code family the spreading procedure introduced in Section II-B is applied to the base matrices $\mathbf{B}^{(k)}$. This can be done in various ways and only minor constraints have to be taken into account. We consider the rate-compatible family of raptor-like LDPC codes $\{\mathcal{C}_k\}_1^K$ with associated base matrices $\mathbf{B}^{(k)}$ from [9] with no further optimization as the origin for our code design. The edge spreading procedure must now be carried out for every base matrix of the protograph code family and for the rest of the paper we consider $m_s = 1$. As shown in (1), the subsequent base matrices are nested – a property that simplifies the spreading process. The procedure is as follows. The precode $\mathbf{B}^{(1)}$ denoted by

$$\mathbf{B}^{(1)} = \begin{bmatrix} 4 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix} \tag{4}$$

is exemplary spread over $m_s + 1 = 2$ time instants into the base matrices $\mathbf{B}_0^{(1)}$ and $\mathbf{B}_1^{(1)}$ as

$$\mathbf{B}_0^{(1)} = \begin{bmatrix} 3 & 0 & 1 & 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{5}$$

$$\mathbf{B}_1^{(1)} = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 2 & 0 & 1 & 1 \end{bmatrix}. \tag{6}$$

As the extension of the original raptor-like LDPC code was accomplished by simply adding a row $[\mathbf{B}^{(e,k+1)}\ \mathbf{0}_{1\times k-2}\ 1]$ to $\mathbf{B}^{(k)}$ denoting the LT code connections, this procedure can be adopted for the spreading procedure. To obtain a new incremental redundancy step k, the row

$$[\mathbf{B}^{(e,k)}\ \mathbf{0}_{1\times k-2}\ 1] \tag{7}$$

is spread into

$$[\mathbf{B}_0^{(e,k)}\ \mathbf{0}_{1\times k-2}\ 1] \tag{8}$$

$$[\mathbf{B}_1^{(e,k)}\ \mathbf{0}_{1\times k-2}\ 0]. \tag{9}$$

Note that the single entry rightmost in the extension vector (denoting the single edge between check node and variable node of the incremental redundancy part in step k), is randomly assigned to a specific time instant. An example spread for incremental redundancy step $k = 2$ is shown below where

$$[\mathbf{B}^{(e,2)}\ 1] = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1] \tag{10}$$

is arbitrarily spread to

$$[\mathbf{B}_0^{(e,2)}\ 1] = [1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1]$$
$$[\mathbf{B}_1^{(e,2)}\ 0] = [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0]$$

The complete spreading of the extension vectors can be seen in (14) on the next page. The resulting rate-compatible LDPC convolutional code base matrix for incremental redundancy step k is then given by

$$\mathbf{B}_{[0,L-1]}^{(k)} = \begin{bmatrix} \mathbf{B}_0^{(k)} & & \\ \mathbf{B}_1^{(k)} & \ddots & \\ & & \mathbf{B}_0^{(k)} \\ & \ddots & \mathbf{B}_1^{(k)} \end{bmatrix} \tag{11}$$

with

$$\mathbf{B}_i^{(k)} = \begin{bmatrix} \mathbf{B}_i^{(k-1)} & & \mathbf{0}_{M_1+k-1\times 1} \\ \mathbf{B}_i^{(e,k)} & \mathbf{0}_{1\times k-2} & 1 \end{bmatrix};\ k = 2,\dots,K \tag{12}$$

for $i = 0, 1$. In the preceding example a spreading over $m_s + 1 = 2$ time instants is shown while other arbitrary spreadings with $m_s > 1$ are possible as long as (3) is fulfilled. The design rate of the rate-compatible convolutional code ensemble for incremental redundancy step k with M_k check nodes and N_k variable nodes is given by

$$R_k = 1 - \left(\frac{L + m_s}{L}\right)\frac{M_k}{N_k} = 1 - \left(\frac{L + m_s}{L}\right)\frac{M_1 + k}{N_1 + k} \tag{13}$$

which introduces a rate loss that vanishes with increasing L. Special attention must be drawn to rows with low check degree as the spreading might introduce checks of degree one that must be avoided. Such rows should not be spread and remain in one time instant introducing an additional rate loss at the boundary which also vanishes with L.

IV. Ensemble analysis

A. Thresholds

To evaluate the performance of the constructed code family under belief propagation decoding, density evolution was applied to the protographs of the individual incremental redundancy steps.

The investigated spreading for incremental redundancy steps $k = 2 \dots 11$ is shown in (14) while assuming the base matrix of the precode for step $k = 1$ to be spread according to (4), (5) and (6). Density evolution was applied to the protograph of every single incremental redundancy step to obtain the channel threshold for the specific ensemble. Additionally these thresholds were calculated for different termination lengths L and the analysis was carried out on the BEC. For comparison the incremental redundancy family based on regular LDPC codes [1] is also considered.

$$
\begin{bmatrix} \mathbf{B}_0^{(e,2)} \\ \mathbf{B}_0^{(e,3)} \\ \vdots \\ \mathbf{B}_0^{(e,11)} \end{bmatrix} = \begin{bmatrix} 1&0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&0&0 \\ 1&0&0&0&0&1&0&1 \\ 0&1&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{B}_1^{(e,2)} \\ \mathbf{B}_1^{(e,3)} \\ \vdots \\ \mathbf{B}_1^{(e,11)} \end{bmatrix} = \begin{bmatrix} 0&1&0&1&0&1&0&1 \\ 0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0&1 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&0&1&0&0&1 \\ 0&0&0&1&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \tag{14}
$$

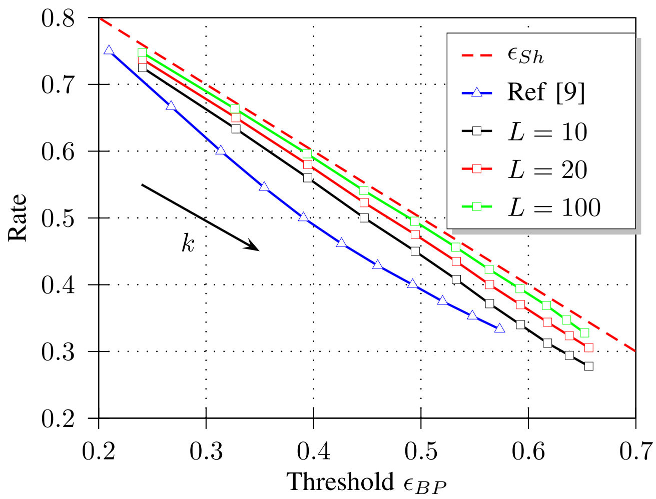

Fig. 4. Threshold/rate plane for coupled rate-compatible LDPC code ensemble on the BEC (Shannon limit ϵ_{Sh})

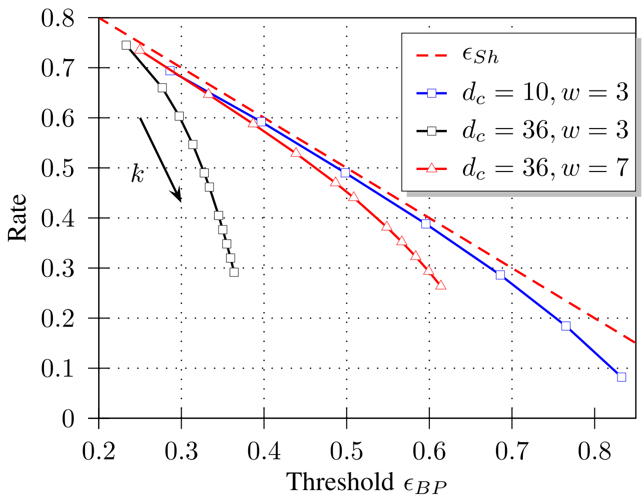

Fig. 5. Threshold/rate plane for regular LDPC convolutional codes with nested incremental redundancy from [1] on the BEC (Shannon limit ϵ_{Sh})

Fig. 4 shows the thresholds for the rate-compatible protograph based convolutional LDPC code family for the BEC. Clearly the threshold improvement for increasing L can be recognized. The uncoupled threshold curve shows a significant gap to the Shannon limit that vanishes when spatial coupling is applied. The coupled incremental redundancy family obtains an almost uniform gap to the Shannon limit.

The thresholds for the comparable regular LDPC convolutional code family were obtained using density evolution on the ensemble in [1] and are shown in Fig. 5 for different check degrees d_c. The code family with $d_c = 10$ was chosen according to its very good performance close to capacity but the rates obtained in the regular case differed significantly from those of the proposed rate-compatible LDPC convolutional code ensemble. Therefore a second family of regular codes was analyzed with higher check degree of $d_c = 36$ and different smoothing parameters w [1]. This family approximates the rates of our incremental redundancy family very well but the performance is decreased due to larger check degree. To overcome this issue the smoothing parameter w can be increased resulting in an improved performance. For $w \geq 7$ the performance of the code family is approaching the Shannon limit but exhibits a larger rate loss as compared to smaller w. It is also desirable to have small memory m_s (respectively w) because this parameter directly influences the complexity of a practically attractive windowed belief propagation decoder [14].

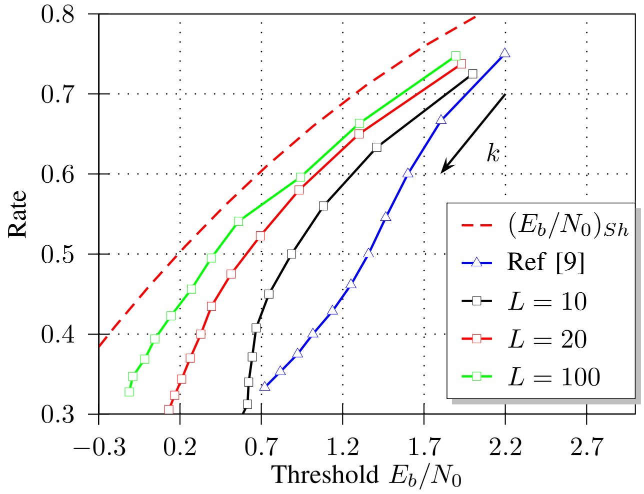

Fig. 6. Threshold/rate plane for coupled rate-compatible LDPC code ensemble on the AWGN channel with Shannon limit $(E_b/N_0)_{Sh}$

The thresholds for the proposed family of rate-compatible codes on the AWGN channel are shown in Fig. 6. For the uncoupled case, the gap to the Shannon limit increases with k. But as spatial coupling is applied, the effect of threshold improvement moves the threshold curves towards the Shannon

limit. For sufficiently large L the gap becomes almost equal at every k and the threshold curves seem to follow the shape of the Shannon limit.

B. Decoding Complexity

The threshold improvement for all incremental redundancy steps k of the raptor-like convolutional code ensemble is of great interest but one should keep in mind at which price such improved thresholds are achieved. We assume the decoding of the constructed code family with the belief propagation algorithm. The complexity of such a message passing algorithm is essentially determined by the number of edges in the corresponding Tanner graph. A measure for the complexity based on this assumption was introduced in [15] and is denoted as $C^{\text{LDPC}} = \bar{d}_v/R$ where $\bar{d}_v$ is the average variable degree (or column weight) of the code and R is the rate. This quantity describes the message calculation operation per information bit. The complexity of the proposed rate-compatible LDPC convolutional coding scheme together with the regular ensembles for different check degrees d_c is shown in Fig. 7. For lower rates the complexity increases as the code introduces more redundancy. But for the regular case the complexity appears to increase almost exponentially for lower rates which can especially be seen for $d_c = 10$. A similar behaviour is obtained for the regular incremental redundancy family with $d_c = 36$. The reason for this is the increasing variable degree d_v at each step k resulting in a uniformly increasing average variable degree. For the proposed rate-compatible LDPC convolutional codes, the complexity increases almost linearly in the range of considered k – an effect that is due to the irregular edge connections. These irregular connections are the key to a flexible trade-off between performance and parametrizable complexity increase.

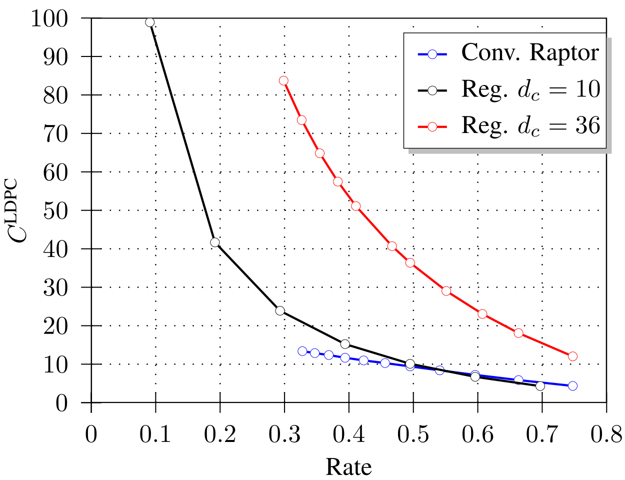

Fig. 7. Decoding complexity C^{LDPC} of proposed ensemble and comparing regular ensembles [1]

V. Conclusion

This paper proposes a class of protograph based rate-compatible LDPC convolutional codes with good thresholds on the BEC and AWGN channel. At lower rates the uncoupled code families from [9] show an increasing gap to the Shannon limit and to reduce this gap spatial coupling was applied. A spreading procedure was introduced to obtain the rate-compatible LDPC convolutional code ensemble. A threshold analysis with density evolution confirmed the expected threshold improvement. Additionally, the code families seem to achieve an almost uniform gap to the Shannon limit. Comparable spatially coupled regular incremental redundancy families also show good thresholds for low check degrees with moderate memory but the complexity may be an issue in the regular setting especially at lower rates. The irregular edge connections of the proposed rate-compatible LDPC convolutional codes help to keep the complexity growth moderate for lower rates which is beneficial in terms of implementation of efficient decoders.

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