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Asymptotic Performances of Woven Graph Codes

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Abstract—Constructions of woven graph codes based on constituent block and convolutional codes are studied. It is shown that within the random ensemble of such codes based on s-partite, s-uniform hypergraphs, where s depends only on the code rate, there exist codes satisfying the Varshamov-Gilbert (VG) and the Costello lower bound on the minimum distance and the free distance, respectively.

Index terms—Graphs, girth, hypergraphs, convolutional codes, graph codes, LDPC codes, woven codes, tailbiting codes.

I. INTRODUCTION

Woven graph codes can be considered as a generalization of low-density parity-check (LDPC) block codes [1]. Their structure as graph codes makes them suitable for iterative decoding. Moreover, the LDPC block codes are known as codes with low-complexity decoding and they can be considered as competitors to the turbo codes [2] which are sometimes called parallel concatenated codes. As mentioned in [3], the underlying graph defines a permutation of the information symbols which resembles the interleaving in turbo coding schemes.

In the sequel we distinguish between graph, graph-based, and woven graph codes. We say that a graph code is a block code whose parity-check matrix coincides with the incidence matrix of the corresponding graph. Graph-based codes is a class of concatenated codes with constituent block codes concatenated with a graph code (see, for example, [3]). Each vertex in a graph corresponds to a constituent block code. The block length of their constituent block codes coincides with the degree of the underlying graph.

We introduce woven graph codes which are, in fact, graph-based codes with constituent block codes whose block length is a multiple of the graph degree c, that is, their block length is lc, where l is an integer. In particular, when l tends to infinity we obtain convolutional constituent codes.

Distance properties of bipartite graph-based codes with constituent block codes were studied in [3]. It was shown that for some range of rates, random graph-based codes with block constituent codes satisfy the VG bound when the block length of the constituent code tends to infinity. In practice infinite length of constituent code leads to rather long graph-based codes with not only rather high decoding complexity of the iterative decoding procedures but also high encoding complexity.

In this paper, we consider a class of the generalized graph-based codes which we call woven graph codes with constituent block and convolutional codes. They are based on s-partite s-uniform hypergraphs. Notice that graph-based codes with constituent block codes based on hypergraphs were considered in [4], [5]. It is mentioned in [4] that Gallager’s LDPC codes are graph codes over hypergraphs.

We consider first woven graph codes with constituent (lc, lb) block codes. In order to analyze their asymptotic performances we modify the approach in [3] to s-partite s-uniform hypergraphs and constituent (lc, lb) block codes. It is shown that when l grows to infinity in the random ensemble of woven graph codes with binary constituent block codes we can find s ≥ 2 such that there exist codes satisfying the VG lower bound on the minimum distance for any rate.

We also generalize the asymptotic analysis to woven graph codes with constituent convolutional codes. It is shown that when the overall constraint length of the woven graph code tends to infinity in the random ensemble of such convolutional codes we can find s ≥ 2 such that there exist codes satisfying the Costello lower bound on the free distance for any rate.

II. GRAPHS AND CODES

A hypergraph is a generalization of a graph in which the edges are subsets of vertices and may connect (contain) any number of vertices. These edges are called hyperedges. A hypergraph is called s-uniform if every hyperedge has cardinality s or, in other words, connects s vertices. If s = 2 the hypergraph is simply a graph. The degree of a vertex in a hypergraph is the number of hyperedges that are connected to (contain) it. If all vertices have the same degree we say that this is the degree of the hypergraph. The hypergraph is c-regular if every vertex has the same degree c.

Let the set V of vertices of an s-uniform hypergraph be partitioned into t disjoint subsets Vj, j = 1, 2, . . . , t. A hypergraph is said to be t-partite if no edge contains two vertices from the same set Vj, j = 1, 2, . . . , t.

In the sequel we consider s-partite, s-uniform, c-regular hypergraphs. Such a hypergraph is a union of s disjoint subsets of vertices. Each vertex has no connections in its own set and is connected with s − 1 vertices in the other subsets. In Fig. 1 a 3-partite, 3-uniform, 4-regular hypergraph is shown. It contains three sets of vertices. They are shown by triangles, rectangulars, and ovals, respectively. There are no
edges connecting vertices inside any of these three sets. The vertices are connected by hyperedges each of which connects three vertices. For hypergraphs we introduce the notion of 

(\(s, \geq d\))-girth that is the length of the shortest connected subgraph in the hypergraph in which each vertex is incident with at least \(d\) hyperedges. A \(2\)-partite, \(2\)-uniform hypergraph is a bipartite graph. For such a hypergraph the \((2, 2)\)-girth is equal to the girth and a connected subgraph is a cycle.

The utility bipartite graph \([8], [9]\) with \(6\) vertices and \(9\) edges is shown in Fig. 2. It contains a set of \(n = 3\) black and a set of \(n = 3\) white vertices. Each vertex has no connections within its own set and is connected with \(c = 3\) vertices from the other set. The girth of this graph is equal to \(g = 4\).

Consider now a binary graph-based code with constituent block codes determined by a bipartite graph. The \(c = 3\) edges leaving one vertex in this case correspond to a codeword of the constituent \((c, b)\) block code of rate \(R^c = b/c\). The parity-check matrix of the corresponding graph-based code with binary constituent block codes is

\[
H = \begin{pmatrix}
H_1 \\
H_2
\end{pmatrix}
\]

where the parity-check matrix \(H_1\) of size \(n \times nc = 3 \times 9\) has the form

\[
H_1 = \begin{pmatrix}
H^c_{11} & 0 & 0 \\
0 & H^c_{22} & 0 \\
0 & 0 & H^c_{33}
\end{pmatrix}
\]

where \(H^c\) is a size \((c - b) \times c = (3 - b) \times 3\) parity-check matrix of the constituent block code, and \(H_2\) is a size \(n \times nc = 3 \times 9\) parity-check matrix which is the permutation of the columns of \(H_1\) determined by the graph. Notice that in general by choosing \(b < c\) and assigning constituent block codes of different rates \(R^c = b/c\) to the same graph we can obtain graph-based codes of different rates. In general, since in an \(s\)-partite, \(s\)-uniform, \(c\)-regular hypergraph the total number of parity checks is equal to \(sn(c - b)\), then the code rate \(R\) of the graph-based code is

\[
R \geq \frac{n(c - s(c - b))}{nc} = s(R^c - 1) + 1
\]

with equality if and only if all parity-checks are linearly independent. If \(s = 2\) then we get \(R \geq 2R^c - 1\).

In this paper we consider generalized graph-based codes called woven graph codes. The constituent codes used in the woven construction can be chosen as \(R^c = lb/lc\) block codes given by the parity check matrix

\[
H^c = \begin{pmatrix}
H^c_{11} & H^c_{12} & \cdots & H^c_{1c} \\
\vdots & \vdots & \ddots & \vdots \\
H^c_{b1} & H^c_{b2} & \cdots & H^c_{bc}
\end{pmatrix}
\]

where each matrix \(H^c_{ij}\) is an \(l \times l\) binary matrix.

To construct an example of a woven graph code with constituent block codes we use a constituent \((4 \times 3, 4 \times 2)\) linear block code with \(d_{\text{min}} = 3\) determined by the parity-check matrix

\[
H^c = \begin{pmatrix}
H^c_1 & H^c_2 & H^c_3 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}
\]

By searching over all possible permutations of \(H^c_1, H^c_2, H^c_3\) we obtain the following parity-check matrix of a woven graph codes.
code based on the utility graph (1)

\[ H_{wg} = \begin{pmatrix}
H_c^1 & H_c^2 & H_c^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H_c^1 & H_c^2 & H_c^3 & 0 \\
0 & 0 & 0 & 0 & H_c^1 & H_c^2 & H_c^3 \\
H_c^2 & 0 & 0 & 0 & 0 & H_c^1 & H_c^3 \\
H_c^3 & 0 & 0 & 0 & 0 & 0 & H_c^1 \\
0 & H_c^1 & H_c^2 & 0 & 0 & 0 & H_c^3 \\
0 & H_c^1 & H_c^2 & 0 & 0 & 0 & 0
\end{pmatrix} \]  (5)

The matrix (5) describes a (36, 12) linear block code with \( d_{\text{min}} = 10 \).

Let \( G^c(D) \) be a minimal encoding matrix [7] of a rate \( R^c = \frac{b}{c} \), memory \( m \) convolutional code, given in polynomial form

\[ G^c(D) = \begin{pmatrix}
F^c(D) \\
\vdots \\
G_{b/c}^c(D)
\end{pmatrix} \]  (6)

where \( F^c_i(D) = g_{ij}^{(0)} + g_{ij}^{(1)} D + g_{ij}^{(2)} D^2 + \cdots + g_{ij}^{(m)} D^m \)

\( i = 1, 2, \ldots, b \), \( j = 1, 2, \ldots, c \), are binary polynomials such that \( m = \max_i \{ \deg g_{ij}^c(D) \} \). The overall constraint length is \( \nu = \sum_i \max_j \{ \deg g_{ij}^c(D) \} \). The binary information sequence \( w^c(D) = (v_1^c(D), v_2^c(D), \ldots, v_b^c(D)) \) is encoded as

\[ w^c(D) = u^c(D) G^c(D) \]

where \( w^c(D) = (v_1^c(D), v_2^c(D), \ldots, v_b^c(D)) \) is a binary code sequence. Let \( H^c(D) \) denote a parity-check matrix of the same code

\[ H^c(D) = \begin{pmatrix}
h_1^{c(1)}(D) & \cdots & h_1^{c(b)}(D) \\
\vdots & \ddots & \vdots \\
h_c^{c(1)}(D) & \cdots & h_c^{c(b)}(D)
\end{pmatrix} \]  (7)

where \( r = c - b \) is the redundancy of the constituent code.

We denote by \( \mathbb{F}_2((D)) \) the field of binary Laurent series and regard a rate \( R^c = \frac{b}{c} \) constituent convolutional code as a rate \( R^c = \frac{b}{c} \) block code \( G^c \) over the field of binary Laurent series encoded by \( G^c(D) \). Then its codewords \( w^c(D) \) are elements of \( \mathbb{F}_2((D))^\nu \), which is the \( \nu \)-dimensional vector space over the field of binary Laurent series [7].

Representing a convolutional code as a block code over the field of binary Laurent series, we can obtain a woven graph code with constituent convolutional codes as a generalization of a graph-based code with binary constituent block codes. The corresponding woven graph code based on an \( s \)-partite, \( s \)-uniform, \( c \)-regular hypergraph has rate \( R = s(R^c - 1) + 1 \) and its constraint length is at most \( s \nu R^c \).

For example, a parity-check matrix \( H_{wg}(D) \) of the rate \( R = 4/3 - 1 = 1/3 \) woven graph code based on the utility graph with \( R^c = 2/3 \) constituent convolutional codes has the form

\[ H_{wg}(D) = \begin{pmatrix}
h_1^c & h_2^c & h_3^c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h_1^c & h_2^c & h_3^c & 0 \\
0 & 0 & 0 & 0 & h_1^c & h_2^c & h_3^c \\
h_2^c & 0 & 0 & 0 & h_1^c & h_3^c & 0 \\
h_3^c & 0 & 0 & 0 & 0 & h_1^c & h_2^c \\
0 & h_1^c & h_2^c & 0 & 0 & 0 & h_3^c \\
0 & h_1^c & h_2^c & 0 & 0 & 0 & 0
\end{pmatrix} \]  (8)

where \( h_1^c \) is short-hand for \( h_1^{c(1)}(D) \), \( h_2^c(D) = (h_1^{c(1)}(D), h_2^{c(1)}(D), h_3^{c(1)}(D)) \) is a parity-check matrix of the rate \( R^c = 2/3 \) constituent convolutional code, \( h_1(D) = 1 + D + D^4 \), \( h_2(D) = 1 + D + D^3 + D^4 + D^5 \), \( h_3(D) = 1 + D^2 + D^3 + D^4 + D^5 \).

Using the BEAST algorithm [11] we found that this rate \( R = 1/3 \) woven graph convolutional code has the free distance 30 and spectrum 4, 0, 0, 0, 0, 3, 0, 6, 0, \ldots. Its overall constraint length is \( \nu = 25 \).

We can regard the \( n \) white constituent convolutional codes as a warp with \( nc \) threads. Each of the \( n \) black constituent convolutional codes are tacked on \( c \) of the threads in the warp such that each thread is tacked on exactly once. Thus, our construction is a special case of a woven code [10] and we call this graph-based code woven graph code.

III. ASYMPTOTIC BOUNDS ON THE MINIMUM AND FREE DISTANCES OF WOVEN GRAPH CODES

We will show that the ensemble of random woven graph codes based on random \( s \)-partite, \( s \)-uniform, \( c \)-regular hypergraphs with a fixed degree \( c \) and a number of vertices \( n \) in each subgraph contains asymptotically good codes. In order to prove this we will modify the approach in [3].

A. Woven graph codes with constituent block codes

First we consider the ensemble of random woven graph codes with rate \( R^c = \frac{b}{c} \) constituent block codes determined by the edges of a random \( s \)-partite, \( s \)-uniform, \( c \)-regular hypergraph corresponding to the time-varying random parity-check matrix

\[ H_{wg} = \begin{pmatrix}
\tilde{H}_1 \\
\vdots \\
\tilde{H}_s
\end{pmatrix} = \begin{pmatrix}
\pi_1(H_1) \\
\vdots \\
\pi_s(H_s)
\end{pmatrix} \]  (9)

where \( \pi_i = \pi_i(H_i) \), \( i = 1, 2, \ldots, s \), is a matrix of size \( nc(1 - R^c) \times nc \) and \( \pi_i \) denotes a random permutation of the columns of \( H_i \),

\[ \tilde{H}_i = \begin{pmatrix}
H_i^{c(1)} & 0 & \cdots & 0 \\
0 & H_i^{c(2)} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_i^{c(n)}
\end{pmatrix} \]  (10)

where \( H_i^{c(t)}, t = 1, \ldots, n \), denotes the random parity-check matrix \( (4) \) which determines the \( (l_c, l_b) \) constituent block code and \( n \) is the number of constituent codes in each subgraph.

Next we prove the following theorem.

**Theorem 1:** (Varshamov-Gilbert lower bound) For any \( \epsilon > 0 \), some \( l_0 > 0 \), some integer \( s > 0 \) and for all \( l > l_0 \) in the random ensemble of length \( nc(l) \) woven graph codes with \( (l_c, l_b) \) binary block constituent codes of rate \( R^c = \frac{b}{c} \) there exist codes of rate \( R_{wg} = s(R^c - 1) + 1 \) such that their relative minimum distance \( \delta_{wg} = d_{\text{min}}/nc \) satisfies the inequalities

\[ \delta \geq \begin{cases}
\delta(R) - \epsilon, & \text{if } R > \log_2(2(1 - \delta_{VG}(R))^s) \\
\delta_{VG}(R) - \epsilon, & \text{if } R \leq \log_2(2(1 - \delta_{VG}(R))^s)
\end{cases} \]  (11)
where \( \delta(R) \) is a root of the equation
\[
(1-s)h(\delta) - \delta s \log_2 \left( 2^{-(R-1)/s} - 1 \right) = 0
\]
\( \delta_{VG} \) is the solution of \( h(\delta) + R - 1 = 0 \), and \( h(\cdot) \) denotes the binary entropy function.

**Proof.** Let \( w \) be the Hamming weight of the codeword \( v \) of the random binary woven graph code \( C_2(H) \). We are going to find the parameter \( d \) such that the probability \( P(v|w) \) tends to 0 for all \( w < d \). We can rewrite \( P(v|w) \) as
\[
P(v|w) = \sum_j P(v|w,j) P(j|w)
\]
where \( j = (j_1, j_2, \ldots, j_s) \) and \( j_i \) denotes the number of nonzero constituent codewords in the \( i \)th subgraph corresponding to the codeword of weight \( w \).

In the ensemble of random parity-check matrices \( H^{c(t)}_t \), \( t = 1, 2, \ldots, n \), of size \( lc(1-R^c) \times lc \) the probability that a nonzero vector \( v^c \) is a codeword of the corresponding constituent random binary code \( C_2[H^c] \) is equal to \( 2^{-(c-b)l} \) since the syndromes of the constituent codes are equiprobable sequences of length \( (c-b)l \). Taking into account that in the \( i \)th subgraph we have \( j_i \) nonzero constituent codewords the probability \( P(v|w,j) \) can be upper-bounded by
\[
P(v|w,j) \leq \left( \frac{nc}{w} \right) \prod_{i=1}^s 2^{-j_i c(1-R^c)}.
\]
From [12] we have the following upper bound on the probability \( P(j|w) \) that a codeword of weight \( w \) contains \( j = (j_1, j_2, \ldots, j_s) \) nonzero constituent codewords in the subgraphs
\[
P(j|w) \leq \prod_{i=1}^s \left( \frac{n}{j_i} \right) \left( \frac{cl}{w/j_i} \right)^{j_i} \left( \frac{w-1}{j_i-1} \right).
\]
Then we have
\[
P(v|w) \leq \sum_j \left( \frac{nc}{w} \right)^{1-s}
\times \prod_{i=1}^s 2^{-j_i c(1-R^c)} \left( \frac{n}{j_i} \right)^{j_i} \left( \frac{cl}{w/j_i} \right)^{j_i} \left( \frac{w-1}{j_i-1} \right)
\leq (n+1)^s \left( \frac{nc}{w} \right)^{1-s}
\times \left( \max_j \left( 2^{-j c(1-R^c)} \left( \frac{n}{j} \right)^{j} \left( \frac{cl}{w/j} \right)^{j} \left( \frac{w-1}{j-1} \right) \right) \right)^s.
\]
Consider the asymptotic behaviour of (12) when \( m \) tends to infinity. Introduce the notations \( \gamma = j/n \) and \( \delta = w/(nc) \) and the function
\[
F(\delta) = \lim_{t \to \infty} \frac{\log_2 P(v|w) = 0|w)}{nc}.
\]
After simple derivations we obtain
\[
F(\delta) \leq \hat{F}(\delta) = (1-s)h(\delta) - (1-R)\gamma + s\gamma h(\delta) + \gamma(\delta - 1)
\]
where \( R = s(R^c - 1) + 1 \) is the rate of binary woven graph code. Maximizing (16) over \( 0 < \gamma \leq 1 \) gives
\[
\gamma_{opt} = \min \left\{ 1, \frac{\delta}{\delta_0} \right\}
\]
where \( \delta_0 = 1 - 2^{(R-1)/s} \). By inserting \( \gamma_{opt} < 1 \) and \( \gamma_{opt} = 1 \) into (16) we obtain
\[
\hat{F}(\delta) = \left\{ \begin{array}{ll}
(1-s)h(\delta) - \delta s \log_2 \left( \frac{1-\delta}{\delta_0} \right), & \text{if } \delta \geq \delta_0 \\
(1-s)h(\delta) - \delta s \log_2 \left( \frac{1-\delta}{\delta_0} \right), & \text{if } \delta < \delta_0
\end{array} \right.
\]
which coincides with (9), (10) in [3] for \( s = 2 \), that is, if the graph is bipartite.

For any \( R \) and \( \delta \) from \( \hat{F}(\delta) < 0 \) it follows that there exist codes of rate \( R \) with relative minimum distance \( \delta \). Let \( \delta(R) \) denote the solution of the equation
\[
\hat{F}(\delta) = 0
\]
for \( 0 < \delta \leq 1 - 2^{(R-1)/s} \) and let \( \delta_{VG} \) be the solution of \( h(\delta) + R - 1 = 0 \). Solving (18) for \( \gamma_{opt} < 1 \) and \( \gamma_{opt} = 1 \) we obtain that there exist codes of rate \( R \) with the relative minimum distance \( \delta \) satisfying the inequality (11). 

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**Fig. 3.** The relative minimum distance as a function of the code rate of the binary woven graph code with block constituent codes.

In Fig. 3 the relative minimum distance of the binary woven graph code with block constituent codes as a function of the code rate is shown. It is easy to see that when \( s \) grows the ensemble of binary woven graph codes contains codes meeting the VG bound for almost all rates \( 0 \leq R \leq 1 \). Fig. (4) demonstrates the gap \( R_{VG} - R \) between the VG bound and the code rate as a function of the relative minimum distance \( \delta \) for different values of \( s \). It follows from Fig. (4) that for \( s \geq 3 \) the difference in code rate compared to the VG bound is negligible.

**B. Asymptotic bound on the free distance of woven graph codes with constituent convolutional codes**

Consider a ZT convolutional woven graph code with constituent ZT convolutional codes of rate \( R^c = b/c \). The length
of a ZT woven graph codeword in $nc$-tuples is equal to $l + m_{wg}$ where $l$ is the number of $nc$-tuples influenced by information symbols and $m_{wg}$ is the memory of the woven graph code of rate $R_{wg} = s(R^c - 1) + 1$. Denote by $d_{\text{free}}$ the free distance of the corresponding woven graph code.

Now we can prove the following

**Theorem 2:** (Costello lower bound) For any $\epsilon > 0$, some $m_0 > 0$, some integer $s \geq 2$, and for all $m_{wg} > m_0$ in the random ensemble of rate $R_{wg} = s(R^c - 1) + 1$ woven graph codes over $s$-partite, $s$-uniform, $c$-regular hypergraphs with constituent convolutional codes of rate $R^c = b/c$ there exists a code with memory $m_{wg}$ such that its relative free distance $\delta_{\text{free}} = d_{\text{free}}/ncm_{wg}$ satisfies the Costello lower bound [7],

$$\delta_{\text{free}} \geq \frac{R_{wg}}{\log_2 (2^{1/R_{wg}} - 1)} - \epsilon. \tag{19}$$

**Proof:** Analogously to the derivations in the proof of Theorem 1 let $j = (j_1, j_2, \ldots, j_s)$ where $j_i$ denotes the number of nonzero constituent codewords in the $i$th subgraph corresponding to the codeword of weight $w$, $j_i \in \{1, \ldots, n\}$. In order to evaluate the number of nonzero constituent codewords among the $n$ constituent codewords, notice that the set of such codewords is a union of sets of nonzero constituent codewords belonging to each of the $s$ subgraphs. The cardinality of the union is at least $j_{\text{max}} = \max_{i=1}^s \{j_i\}$. Therefore the all-zero “tail” required to force the encoder into the zero state has length at least $j_{\text{max}} cm_{wg}$. The total number of redundant symbols consists of two parts: the number $\sum_{i=1}^s j_i cl(1-R^c)$ of parity-check symbols for the nonzero constituent codewords in the $s$ subgraphs and at least $j_{\text{max}} cm_{wg}$ redundant symbols required for zero-tail terminating of the woven graph code. Making the corresponding changes in (13) and (15) and introducing the notations $\delta = w/cm_{wg}$, $\mu = l/m_{wg}$, $\gamma = j/n$, we obtain

$$F(\delta) = \lim_{m_{wg} \to \infty} \log_2 P(\mathbf{w}^T = 0|\mathbf{w}) \frac{ncm_{wg}}{m_{wg}}$$

\[\leq \max_{\gamma \in (0,1)} \left( (1-s)(1+\mu)h \left( \frac{\delta}{1+\mu} - \frac{\gamma}{\gamma(1+\mu)} \right) \right) + \gamma(1+\mu)sh \left( \frac{\delta}{\gamma(1+\mu)} \right). \tag{20}\]

Maximizing (20) over $0 < \gamma \leq 1$, we obtain that if $s$ is large enough, then $\gamma_{\text{opt}} = 1$. It follows from (20) that

$$F(\delta) \leq (1+\mu)h \left( \frac{\delta}{1+\mu} \right) - 1 - \mu + \mu R_{wg}. \tag{21}$$

Maximization of $F(\delta)$ over $\mu$ gives

$$F_{\text{opt}}(\delta) = -\delta \log_2 (2^{1/R_{wg} - 1}) - R_{wg} \tag{22}$$

where

$$\mu_{\text{opt}} = \frac{\delta}{1 - 2^{1/R_{wg} - 1}} - 1.$$

We can find a bound on $\delta_{\text{free}}$ by solving $F_{\text{opt}}(\delta) = 0$. Thus, we can conclude that for any $\epsilon > 0$ we can find a woven graph code such that (19) holds.

**IV. Conclusion**

The asymptotic behaviour of the woven graph codes with block as well as with convolutional constituent codes has been studied. It was shown that in the random ensemble of such codes based on $s$-partite, $s$-uniform, $c$-regular hypergraphs we can find a value $s \geq 2$ such that there exist codes meeting the VG and the Costello lower bound on the minimum distance and free distance, respectively, for any code rate.

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