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Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

Gerhard Kristensson
Abstract

This paper analyzes and solves an integral and its indefinite Fourier transform of importance in multiple scattering problems of randomly distributed scatterers. The integrand contains a radiating spherical wave, and the two-dimensional domain of integration excludes a circular region of varying size. A solution of the integral in terms of radiating spherical waves is demonstrated. The method employs the Erdélyi operators, which leads to a recursion relation. This recursion relation is solved in terms of a finite sum of radiating spherical waves. The solution of the indefinite Fourier transform of the integral contains the indefinite Fourier transforms of the Legendre polynomials, which are solved by a closed formula.

1 Introduction

In recent years, the electromagnetic scattering problem by randomly distributed objects has been successfully formulated and solved. Some important contributions in the field are found in e.g., [3–8, 10, 11, 13, 16–19, 21–25]. These references refer to various aspects of the topic, and more references can be found in these papers. The topic is also treated in several textbooks, see e.g., [12, 14, 20], which can be consulted for a comprehensive treatment of the various multiple scattering theories.

Of critical importance for the solution of a specific scattering problem with hole-corrections (HC) is an integral of the form [9, 18, 20]

\[ I_l(z) = \frac{k^2}{2\pi} \int \int_{\mathbb{R}^2} H(r - a)h_1^{(1)}(kr)P_l(\cos \theta)\, dx\, dy, \quad z \in \mathbb{R} \]

where \( H(x) \) denotes the Heaviside function, \( h_1^{(1)}(kr) \) the spherical Hankel function, and \( P_l(x) \) the Legendre polynomial of order \( l \), respectively. We have also adopted the spherical coordinates, \( r = \sqrt{x^2 + y^2 + z^2} \) and \( \theta (\cos \theta = z/r) \), and the wave number \( k \). The domain of integration is the plane \( z = \) constant, excluding the sphere of radius \( a > 0 \) at the center, see Figure 1. For a given value of \( |z| \leq a \), the radius of the excluded circle is \( \sqrt{a^2 - z^2} \). For \( |z| \geq a \) the integration is the entire \( x-y \) plane. This integral, for a given \( a > 0 \), is a non-trivial function of \( z \in \mathbb{R} \). To ensure convergence of the integral at infinity, we assume the wave number \( k \) has an arbitrarily small imaginary part. The explicit solution of this integral, as a function of \( z \) and the index \( l = 0, 1, 2, \ldots \), is the aim of this paper, and the goal is to express the solutions in a form that is attractive from a numerical computation point of view.

The solution of the integral \( I_l(z) \) is developed in Sections 2 and 3. The indefinite Fourier transform of \( I_l(z) \) is also essential for a successful solution of the multiple scattering problem with hole-corrections, and this analysis is found in Sections 4 and 5. The paper is concluded with a short summary in Section 6.
Figure 1: The geometry of the integration domain — the plane $z = \text{constant}$ (dotted line), and the exclusion volume — the sphere of radius $a$ located at the origin (in gray).

2 The integral $I_l(z)$

Rewrite the integral $I_l(z)$ in (1.1) in cylindrical coordinates and perform the integration in the azimuthal angle. We get from (1.1)

$$I_l(z) = k^2 \int_{h(z)}^{\infty} h_l^{(1)} \left( k \sqrt{\rho^2 + z^2} \right) P_l \left( z/\sqrt{\rho^2 + z^2} \right) \rho \, d\rho, \quad z \in \mathbb{R}$$

where

$$h(z) = \begin{cases} \sqrt{a^2 - z^2}, & -a \leq z \leq a \\ 0, & |z| > a \end{cases}$$

From the parity of the Legendre polynomials, $P_l(-x) = (-1)^l P_l(x)$, we see that also $I_l(-z) = (-1)^l I_l(z)$. Thus, it suffices to evaluate the integral for $z > 0$. In particular, $I_l(0) = 0$, if $l$ is an odd integer. From (2.1) we also easily compute the integral for $l = 0$, viz.,

$$I_0(z) = \begin{cases} e^{-ikz}, & z \leq -a \\ ikah_0^{(1)}(ka) = e^{ika}, & -a \leq z \leq a \\ e^{ikz}, & z \geq a \end{cases}$$

2.1 Solution outside the interval $[-a, a]$

In the interval $z > a$, the integral is evaluated with the use of the transformation of the outgoing scalar spherical wave in terms of planar waves [2, p. 180], i.e., for a
general value of $z \neq 0$

\[
h_i^{(1)} \left( k \sqrt{\rho^2 + z^2} \right) \frac{P_l(\pm|z|/\sqrt{\rho^2 + z^2})}{k} \frac{dk_x dk_y}{k^2}, \quad z \geq 0
\]

where $\rho = x \hat{x} + y \hat{y}$, $k_t = k_z \hat{x} + k_y \hat{y}$, $k_t = |k_t|$, and $k_z$ is defined by

\[
k_z = (k^2 - k_t^2)^{1/2} = \begin{cases} \sqrt{k^2 - k_t^2} & \text{for } k_t < k \\ i\sqrt{k_t^2 - k^2} & \text{for } k_t > k \end{cases}
\]

For $z > a$, we get from (1.1)

\[
I_l(z) = \frac{k^2}{2\pi} \int_{\mathbb{R}^2} \frac{i^{-l}}{2\pi} \left( \int_{\mathbb{R}^2} P_l(k_t/k) e^{ik_t \rho + ik_z |z|} \frac{k}{k_z} \frac{dk_x dk_y}{k^2} \right) dx \, dy
\]

by orthogonality or completeness of the planar waves.\(^1\) This implies that the integral for $z > a$ is

\[
I_l(z) = i^{-l}e^{ikz}, \quad z > a
\]

and consequently, by parity, or analogous calculations

\[
I_l(z) = i^{l}e^{-ikz}, \quad z < -a
\]

We observe that the integral outside the interval $[-a, a]$ is not singular as $a \to 0$. In fact, the module is constant 1.

3 Solution of the integral $I_l(\eta)$, $-a \leq z \leq a$

We have already obtained a solution of the integral in the interval $|z| > a$, and we now concentrate on finding a solution of the integral in the interval $-a \leq z \leq a$.

The Erdélyi operators $Y_n^m$ in Ref. 12 are instrumental in finding a closed formula for the integral $I_l(z)$. From [12, Th. 3.13], we have the following very useful result:

\[
D \left( h_i^{(1)}(kr)P_l(\cos \theta) \right) = \frac{l + 1}{2l + 1} h_{l+1}^{(1)}(kr)P_{l+1}(\cos \theta) - \frac{l}{2l + 1} h_{l-1}^{(1)}(kr)P_{l-1}(\cos \theta)
\]

where $D = -k^{-1}(\partial/\partial z)$. The $D$ operator and the Erdélyi operators are related by $Y_0^\phi = \sqrt{\frac{2}{4\pi}} D_1^0 = \sqrt{\frac{2}{4\pi}} D$.

\(^1\)To ensure convergence of the integral at infinity, assume the wave number $k$ has an arbitrary small, positive imaginary part.
Apply the differential operator $D$ to the integral $I_l(z)$ in (2.1), and use the relation above. We obtain, since $h'(z)h(z) = -z$, the following recursion relation:

$$DI_l(z) = -kzh_l^{(1)}(ka)P_l(z/a) + \frac{l+1}{2l+1}I_{l+1}(z) - \frac{l}{2l+1}I_{l-1}(z), \quad -a \leq z \leq a$$

with initial condition $I_0(z) = ikh_0^{(1)}(ka)$.

In the dimensionless variables $\eta = kz$ and $\xi = ka > 0$, this leads to the recursion relation,

$$l = 0, 1, 2, \ldots \quad (3.1)$$

The recursion relation is conveniently put in a more generic form by introducing the variable $x = \eta/\xi \in [-1, 1]$. The dependent variable is now $x$, and $\xi$ is a parameter. Retaining the same notation for the integral, but with a change of the independent variable, we get

$$I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi) xP_l(x) - \frac{2l+1}{\xi(l+1)}I'_l(x) + \frac{l}{l+1}I_{l-1}(x), \quad -1 \leq x \leq 1$$

The following proposition states the surprisingly simple and elegant solution of this recursion relation.

**Proposition 3.1.** The recursion relation

$$I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi) xP_l(x) - \frac{2l+1}{\xi(l+1)}I'_l(x) + \frac{l}{l+1}I_{l-1}(x), \quad l = 0, 1, 2, \ldots \quad (3.1)$$

with initial condition

$$I_0(z) = i\xi h_0^{(1)}(\xi)$$

has the solution

$$I_l(x) = -\xi h_{l+1}^{(1)}(\xi) P_l(x) + \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \left( \xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi) \right) P_{l-2k}(x), \quad l = 0, 1, 2, \ldots \quad (3.2)$$

\[2\]Outside the interval $z \in [-a, a]$ the recursion relation reads

$$I_{l+1}(z) = \frac{2l+1}{l+1}DI_l(z) + \frac{l}{l+1}I_{l-1}(z), \quad I_0(z) = e^{ikz} \quad z \geq a$$

which is easily solved by induction over the integer $l$. The result is

$$I_l(z) = i^{-l}e^{ikz}, \quad z \geq a$$

in agreement with the result above.
Proof. We prove the proposition by induction over the integer \(l\). The recursion relation (3.2) is true for \(l = 0\), due to the properties of the spherical Hankel functions [15, 10.16.1]. We have from (3.2)

\[
I_0(x) = \xi h_{l-1}^{(1)}(\xi) = \xi \left( \frac{\pi}{2\xi} \right)^{1/2} H_{1/2}^{(1)}(\xi) = i\xi \left( \frac{\pi}{2\xi} \right)^{1/2} H_{1/2}^{(1)}(\xi) = i\xi h_{0}^{(1)}(\xi)
\]

Now assume the solution (3.2) holds for all integers less than or equal to \(l\), and we want to prove that it holds for \(l + 1\). We have from (3.1) and the induction assumption

\[
I_{l+1}(x) = \frac{2l+1}{l+1} \xi h_l^{(1)}(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I_l'(x) + \frac{l}{l+1} I_{l-1}(x)
\]

\[
= \xi h_l^{(1)}(\xi) P_{l+1}(x) + \frac{2l+1}{\xi(l+1)} \xi h_{l+1}^{(1)}(\xi) P_l'(x)
\]

\[
- \frac{2l+1}{\xi(l+1)} \sum_{k=0}^{[l/2]} (-1)^k \left( \xi h_{l+1-2k}(\xi) + \xi h_{l-1-2k}(\xi) \right) P_{l-2k}'(x)
\]

\[
+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left( \xi h_{l-2k}(\xi) + \xi h_{l-2-2k}(\xi) \right) P_{l-2k}'(x)
\]

where we used the following recursion relation for the Legendre polynomials:

\[
(2l+1)x P_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x)
\]

We conclude that \(I_{l+1}(x)\) is a polynomial in \(x\) of the order \(l + 1\), and therefore can be expanded in a series of Legendre polynomials. The form is

\[
I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)
\]

where \(a_n\) depends on \(l\) and \(\xi\). The coefficients \(a_n\) are determined by orthogonality of the Legendre polynomials.

\[
a_n = \frac{2l+3-4n}{2} \int_{-1}^{1} I_{l+1}(x) P_{l+1-2n}(x) \, dx
\]

The first coefficient is special.

\[
a_0 = \xi h_{l}^{(1)}(\xi) = -\xi h_{l+2}^{(1)}(\xi) + \left( \xi h_{l+2}^{(1)}(\xi) + \xi h_{l}^{(1)}(\xi) \right)
\]

Proceed in the same way with the remaining coefficients, \(n = 1, 2, \ldots, [(l+1)/2]\).
where we used the notion

\[ I_{k,n} = \int_{-1}^{1} P'_k(x) P_n(x) \, dx = \begin{cases} 
0, & 0 \leq k \leq n \\
1 - (-1)^{k+n}, & 0 \leq n < k 
\end{cases} \]

Use this result, and the following recursion relation for the spherical Hankel functions:

\[ (2l + 1)h_l^{(1)}(\xi) = \xi h_{l+1}^{(1)}(\xi) + \xi h_{l-1}^{(1)}(\xi) \]  

(3.3)

We get

\[ a_n = \frac{2l + 1}{l + 1} (2l + 3 - 4n) \left( h_{l+1}^{(1)}(\xi) - \sum_{k=0}^{n-1} (-1)^k \left( h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi) \right) \right) \]

\[ + \frac{l}{l + 1} (-1)^{n-1} \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) \]

\[ = \frac{2l + 1}{l + 1} (-1)^n (2l + 3 - 4n) h_{l+1-2n}^{(1)}(\xi) - \frac{l}{l + 1} (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) \]

\[ = (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) \]

Collecting the results gives

\[ I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x) \]

\[ = -\xi h_{l+2}^{(1)}(\xi) P_{l+1}(x) + \sum_{n=0}^{[(l+1)/2]} (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) P_{l+1-2n}(x) \]

which is the statement (3.2) for \( l + 1 \), and the proposition is proved. \( \Box \)

Alternative expressions of the integral \( I(z) \) in the interval \( z \in [-a, a] \) can be found. The following corollary shows some.

**Corollary 3.1.** The integral \( I(z) \) in Proposition 3.1 has the following alternative expressions:

\[ I_l(x) = -\xi h_{l+1}^{(1)}(\xi) P_l(x) + \sum_{k=0}^{[l/2]} (-1)^k (2l - 4k + 1) h_{l-2k}^{(1)}(\xi) P_{l-2k}(x), \quad l = 0, 1, 2, \ldots \]

(3.4)

and

\[ I_l(x) = i^{l-1} \xi h_{l-1}^{(1)}(\xi) P_{l-2l/2}(x) \]

\[ + \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-2k-1}^{(1)}(\xi) (P_{l-2k}(x) - P_{l-2k-2}(x)), \quad l = 0, 1, 2, \ldots \]

(3.5)
and

\[
I_l(x) = i^{l+1} \xi h_0^{(1)}(\xi) P_{2l+1}(x)
- \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{1-2k}^{(1)}(\xi) \frac{2l-4k-1}{(l-2k-1)(l-2k)} P_{2k-1}^{(1)}(x), \quad l = 0, 1, 2, \ldots
\]  

(3.6)

where the two last sums are zero for \( l = 0, 1 \).

**Proof.** The solution in (3.4) is equivalent to (3.2), which is easily seen since the spherical Hankel functions \( h_l^{(1)}(\xi) \) satisfy the recursion relation (3.3). The representation in (3.5) is simply a rearrangement of the sum in (3.2). We obtain from (3.2) \( (l = 0, 1, 2, \ldots) \)

\[
I_l(x) = \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{1-2k}^{(1)}(\xi) (P_{2k}(x) - P_{2+2k}(x))
+ (-1)^{l/2} \xi h_{1-2l/2}^{(1)}(\xi) P_{2l}(x)
= \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{1-2k}^{(1)}(\xi) (P_{2k}(x) - P_{2-2k}(x))
+ i^{l+1} \xi h_0^{(1)}(\xi) P_{2l}(x)
\]

where we used [15, 10.16.1]

\[
h_1^{(1)}(\xi) = i h_0^{(1)}(\xi)
\]

Finally, the relation (3.6) from (3.5) with the use of the recursion relation

\[
l(l + 1) (P_{l+1}(x) - P_{l-1}(x)) = -(2l + 1)(1 - x^2) P_l'(x)
\]

In the original variables \( z \) and \( a \), we have

\[
I_l(z) = i^{l+1} k a h_0^{(1)}(ka) P_{2l+1}(z/a)
+ \sum_{n=0}^{[l/2]-1} (-1)^n k a h_{1-2n}^{(1)}(ka) (P_{2n}(z/a) - P_{2n-2}(z/a)), \quad l = 0, 1, 2, \ldots
\]

or

\[
I_l(z) = -k a h_{1}^{(1)}(ka) P_l(z/a)
+ \sum_{k=0}^{[l/2]} (-1)^k (2l - 4k + 1) h_{1-2k}^{(1)}(ka) P_{2k}(z/a), \quad l = 0, 1, 2, \ldots
\]

and we see that the integral \( I_l(z) \) can be written as a finite sum of spherical waves (except the first term). The most singular term in powers of \( ka \) is of the order \( (ka)^{1-l} \) (order \( O(1) \) if \( l = 0 \)), which is most easily seen from the representation in (3.5).
4 Fourier transform of $I_l(z)$

The indefinite Fourier transform of the function $I_l(z)$ has also importance in the analysis of [9]. More specifically, our goal in this section is to compute

$$\hat{I}_l^+(z) = k \int_{z_0}^z I_l(t)e^{\pm ikt} \, dt, \quad z \geq z_0, \quad l = 0, 1, 2, \ldots$$  \hspace{1cm} (4.1)

where $z_0$ is a fixed number such that $z_0 < -a$.

The function $I_l(t)$ has explicit forms in the three intervals $[z_0, -a]$, $(-a, a)$, and $[a, \infty)$. The explicit forms are:

$$I_l(t) = i^l e^{-ikt}, \quad t \leq -a$$

and in the interval $t \in (-a, a)$ as a finite sum of spherical waves

$$I_l(t) = i^l \left\{ k \left( -a - z_0 \right) \cdot \frac{1}{2i} \left( e^{-2ikz_0} - e^{-2ikz} \right) \right. + \sum_{n=0}^{[l/2]-1} (-1)^n kah_i^{(1)}(ka) (P_{l-2n}(t/a) - P_{l-2n-2}(t/a))$$

In the interval $t \geq a$

$$I_l(t) = i^l e^{ikt}$$

To compute the indefinite Fourier transform we need to calculate the function

$$h_l^+(z) = k \int_{-a}^z P_l(t/a)e^{\pm ikt} \, dt = ka \int_{-1}^{z/a} P_l(t)e^{\pm ikat} \, dt, \quad |z| \leq a$$  \hspace{1cm} (4.2)

For $z = a$ the integral is a spherical Bessel function, viz.,

$$h_l^+(a) = k \int_{-a}^a P_l(t/a)e^{\pm ikt} \, dt = ka \int_{-1}^1 P_l(t)e^{\pm ikat} \, dt = 2ka(\pm i)^{l} j_l(ka)$$

We divide the interval $[z_0, z]$ in three parts. In the interval $z_0 \leq z < -a$, we have

$$\hat{I}_l^+(z) = i^l k \int_{z_0}^z e^{l(\pm 1-1)kt} \, dt = i^l \left\{ \frac{k(z - z_0)}{2i} \left( e^{-2ikz_0} - e^{-2ikz} \right) \right. + \sum_{n=0}^{[l/2]-1} (-1)^n kah_i^{(1)}(ka) h_l^{\pm}(z) \right.$$}

and in the interval $-a < z < a$, we have

$$\hat{I}_l^+(z) = i^l \left\{ \frac{k(-a - z_0)}{2i} \left( e^{-2ikz_0} - e^{2ikz} \right) \right. + \sum_{n=0}^{[l/2]-1} (-1)^n kah_i^{(1)}(ka) h_l^{\pm}(z) \right.$$}

$$+ \sum_{n=0}^{[l/2]-1} (-1)^n kah_i^{(1)}(ka) h_l^{\pm}(z) \right.$$}
and in the interval $a < z$, we have

$$
\hat{I}_l^\pm(z) = i^l \left\{ k(-a - z_0) + \frac{1}{2i} (\cos - 2ika) + i^{1-l} 2(k\alpha)^2 h_0^{(1)}(ka)(\pm i)^{1-l-\frac{1}{2}} j_{l-2\frac{1}{2}}(ka) \right. \\
+ 2(k\alpha)^2(\pm i)^{1-l} \sum_{n=0}^{[l/2]-1} h_{l-2n-1}(ka) (j_{l-2n}(ka) + j_{l-2n-2}(ka)) \\
+ i^{1-l} \left\{ \frac{1}{2i} (e^{2ika} - e^{2ika}) + \frac{1}{2i} (e^{2ika} - e^{2ika}) \right. \\
\left. + 2(k\alpha)^2(\pm i)^{1-l} \sum_{n=0}^{[l/2]-1} h_{l-2n-1}(ka) (j_{l-2n}(ka) + j_{l-2n-2}(ka)) \right\}
$$

5  **Indefinite integral of Legendre polynomials**

It remains to find an effective method to compute the functions $h_l^\pm(z)$ in (4.2). To this end, define

$$
h_l(\eta, \zeta) = \int_{-1}^{\eta} P_l(t) e^{i\zeta t} \, dt, \quad |\eta| \leq 1 \tag{5.1}
$$

We see that $h_l(1, \zeta) = 2i^l j_l(\zeta)$. In terms of the functions $h_l(\eta, \zeta)$, the functions $h_l^\pm(z)$ are

$$
h_l^\pm(z) = k h_l(z/a, \pm ka)
$$

Our ambition in this section is to find an efficient method to compute the integrals in (5.1). We express the function $h_l(\eta, \zeta)$ as a recursion relation.

5.1  **Solution by recursion**

The following recursion relation of Legendre polynomials is useful:

$$
P_l(t) = \frac{1}{2l + 1} \left( P'_{l+1}(t) - P'_{l-1}(t) \right)
$$

Integration by parts then implies ($P_l(-1) = (-1)^l$)

$$
h_l(\eta, \zeta) = \int_{-1}^{\eta} P_l(t) e^{i\zeta t} \, dt = \frac{1}{2l + 1} \int_{-1}^{\eta} \left( P'_{l+1}(t) - P'_{l-1}(t) \right) e^{i\zeta t} \, dt \\
= \frac{1}{2l + 1} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{i\zeta \eta} - \frac{i\zeta}{2l + 1} (h_{l+1}(\eta, \zeta) - h_{l-1}(\eta, \zeta))
$$

or solving for $h_{l+1}(\eta, \zeta)$

$$
h_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{i\zeta \eta} - \frac{2l + 1}{i\zeta} h_l(\eta, \zeta) + h_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \ldots
$$

The functions $h_l(\eta, \zeta)$ can therefore be found by iteration with starting values

$$
h_0(\eta, \zeta) = \frac{1}{i\zeta} (e^{i\zeta \eta} - e^{-i\zeta}) = \frac{1}{i\zeta} P_0(\eta) e^{i\zeta \eta} + h_0^{(2)}(\zeta) = \eta h_0^{(1)}(\zeta \eta) + h_0^{(2)}(\zeta)
$$
and
\begin{align*}
h_1(\eta, \zeta) &= \frac{1}{i\zeta} (\eta e^{i\zeta} + e^{-i\zeta}) + \frac{1}{\zeta^2} (e^{i\zeta} - e^{-i\zeta}) \\
&= \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right) e^{i\zeta} + ih_1^{(2)}(\zeta) = \eta^2 h_1^{(1)}(\eta\zeta) + ih_1^{(2)}(\zeta)
\end{align*}

To find the general solution to this recursion scheme, we start by solving the homogeneous difference equation.

**Lemma 5.1.** The solution to the homogeneous difference equation
\begin{equation*}
a_{l+1} + \frac{2l + 1}{i\zeta} a_l - a_{l-1} = 0, \quad l = 1, 2, 3, \ldots
\end{equation*}
given the initial values \( a_0 \) and \( a_1 \) is
\begin{align*}
a_l &= -\frac{\zeta^2}{2i} \left( a_0 h_0^{(2)}(\zeta) - ia_1 h_0^{(2)}(\zeta) \right) i^l h_l^{(1)}(\zeta) \\
&\quad + \frac{\zeta^2}{2i} \left( a_0 h_0^{(1)}(\zeta) - ia_1 h_0^{(1)}(\zeta) \right) i^l h_l^{(2)}(\zeta), \quad l = 2, 3, 4, \ldots
\end{align*}

**Proof.** Two linearly independent solutions to the homogeneous difference equation in the lemma are \( i^l h_l^{(1)}(\zeta) \) and \( i^l h_l^{(2)}(\zeta) \), which is easily proved by the recursion relation \( f_{l+1}(z) - (2l + 1)f_l(z)/z + f_{l-1}(z) = 0 \), where \( f_l(z) \) is any spherical Bessel or Hankel function. The general solution therefore is
\begin{equation*}
a_l = c_1 i^l h_l^{(1)}(\zeta) + c_2 i^l h_l^{(2)}(\zeta), \quad l = 2, 3, 4, \ldots
\end{equation*}
where \( c_1 \) and \( c_2 \) are constants determined by the starting values \( a_0 \) and \( a_1 \). Explicitly, we get
\begin{equation*}
\begin{cases}
c_1 h_0^{(1)}(\zeta) + c_2 h_0^{(2)}(\zeta) = a_0 \\
c_1 i h_1^{(1)}(\zeta) + c_2 i h_1^{(2)}(\zeta) = a_1
\end{cases}
\end{equation*}
with solution
\begin{equation*}
\begin{cases}
c_1 = -\frac{\zeta^2}{2i} \left( a_0 h_0^{(2)}(\zeta) - ia_1 h_0^{(2)}(\zeta) \right) \\
c_2 = \frac{\zeta^2}{2i} \left( a_0 h_0^{(1)}(\zeta) - ia_1 h_0^{(1)}(\zeta) \right)
\end{cases}
\end{equation*}
where we used the Wronskian of the spherical Hankel functions.
\begin{equation*}
h_n^{(2)}(z) h_n^{(1)}(z)' - h_n^{(2)}'(z) h_n^{(1)}(z) = \frac{2i}{z^2}
\end{equation*}
and
\begin{equation*}
h_0^{(1,2)}(z) = -h_1^{(1,2)}(z). \quad \text{This completes the proof of the lemma.}
\end{equation*}

We are now ready for the solution to the inhomogeneous difference equation in \( h_l(\eta, \zeta) \) above. We formulate this as a lemma.
Lemma 5.2. Define an iteration scheme by
\[ h_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{i\zeta} - \frac{2l + 1}{i\zeta} h_{l}(\eta, \zeta) + h_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \ldots \]
with starting values
\[ h_0(\eta, \zeta) = \eta h_0^{(1)}(\zeta \eta) + h_0^{(2)}(\zeta) \]
and
\[ h_1(\eta, \zeta) = i \left( \eta^2 h_1^{(1)}(\zeta \eta) + h_1^{(2)}(\zeta) \right) \]
The solution is
\[ h_l(\eta, \zeta) = f_l(\eta, \zeta)e^{i\zeta} + i^l h_l^{(2)}(\zeta), \quad l = 0, 1, 2, 3, \ldots \]
where
\[ f_l(\eta, \zeta) = i^l h_l^{(1)}(\zeta) \left\{ \sum_{k=1}^{l} \frac{1}{\zeta^k h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( -\sum_{n=0}^{k} i^{-n+1}(2n + 1) \frac{h_n^{(2)}(\zeta) P_n(\eta)}{\zeta} \right) \right. \]
+ \left. i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \right) - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)}, \quad l = 0, 1, 2, \ldots \]

Proof. We first subtract the part of the solution that contains the spherical Hankel function of the second kind \( h_l^{(2)}(\zeta) \) and the exponential function \( e^{i\zeta} \). To this end, let \( h_l(\eta, \zeta) = f_l(\eta, \zeta)e^{i\zeta} + i^l h_l^{(2)}(\zeta) \). The recursion relation for \( f_l(\eta, \zeta) \) is easily found by the use of the recursion relation \( h_{l+1}^{(2)}(z) = (2l + 1)h_l^{(2)}(z) - h_{l-1}^{(2)}(z) \). We get the new difference equation
\[ f_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) - \frac{2l + 1}{i\zeta} f_l(\eta, \zeta) + f_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \ldots \]
with starting values
\[ f_0(\eta, \zeta) = \frac{1}{i\zeta} P_0(\eta) \]
and
\[ f_1(\eta, \zeta) = \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right) \]
To simplify the notation, we put the difference equation in a standard form [1].
\[ a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = q(n), \quad n = 1, 2, \ldots \]
where
\[
\begin{align*}
  a_n &= f_{n-1}(\eta, \zeta) \\
  p_1(n) &= \frac{2n + 1}{i\zeta} \\
  p_0(n) &= -1 \\
  q(n) &= \frac{1}{i\zeta} (P_{n+1}(\eta) - P_{n-1}(\eta))
\end{align*}
\]
with initial values

\[
\begin{align*}
a_1 &= \frac{1}{i\zeta} P_0(\eta) \\
a_2 &= \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right)
\end{align*}
\]

A solution to the homogeneous difference equation is (see Lemma 5.1)

\[y_l = (1)^{l-1} h_{l-1}^{(1)}(\zeta)\]

The final solution then is \([1], \text{ for } i = 3, 4, \ldots\)

\[a_n = \left( \sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \frac{p_0(j) y_j}{y_{j+2}} \right) \left( \sum_{l=1}^{k-1} \frac{g(l)}{y_{l+2}} \left[ \prod_{m=1}^{l} \frac{p_0(m) y_m}{y_{m+2}} \right]^{-1} + \frac{a_2}{y_2} - \frac{a_1}{y_1} \right) y_n \]

Insert the explicit values, and we obtain

\[f_l(\eta, \zeta) = \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( \sum_{n=1}^{k-1} h_n^{(1)}(\zeta) \frac{P_{n+1}(\eta) - P_{n-1}(\eta)}{i^n} + i h_1^{(1)}(\zeta) P_0(\eta) \\
- h_0^{(1)}(\zeta) \left( P_1(\eta) + i \frac{1}{\zeta} P_0(\eta) \right) \right) - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\} i^l h_l^{(1)}(\zeta), \text{ for } l = 2, 3, 4, \ldots\]

This relation holds also for \(l = 0, 1\), provided the sums with upper limit smaller than the lower limit are interpreted as zero.

We now simplify the sum in this expression.

\[S = - \sum_{n=1}^{k-1} i^{-n} h_n^{(1)}(\zeta) \left( P_{n+1}(\eta) - P_{n-1}(\eta) \right) + i h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta) \]

\[= i h_1^{(1)}(\zeta) (P_2(\eta) - P_0(\eta)) + h_2^{(1)}(\zeta) (P_3(\eta) - P_1(\eta)) - i h_3^{(1)}(\zeta) (P_4(\eta) - P_2(\eta)) + \ldots - i^{-k+2} h_{k-2}^{(1)}(\zeta) (P_{k-1}(\eta) - P_{k-3}(\eta)) - i^{-k+1} h_{k-1}^{(1)}(\zeta) (P_k(\eta) - P_{k-2}(\eta)) + i h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta) \]

\[= \left( h_0^{(1)}(\zeta) + h_2^{(1)}(\zeta) \right) P_1(\eta) + i \left( h_1^{(1)}(\zeta) + h_3^{(1)}(\zeta) \right) P_2(\eta) + \left( h_2^{(1)}(\zeta) + h_4^{(1)}(\zeta) \right) P_3(\eta) - i \left( h_3^{(1)}(\zeta) + h_5^{(1)}(\zeta) \right) P_4(\eta) + \ldots \]

\[+ i^{-k+2} \left( h_{k-2}^{(1)}(\zeta) + h_{k}^{(1)}(\zeta) \right) P_{k-1}(\eta) - i^{-k+1} \left( h_{k-1}^{(1)}(\zeta) + h_{k+1}^{(1)}(\zeta) \right) P_k(\eta) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \]

The recursion relation \(h_{l+1}^{(1)}(z) + h_l^{(1)}(z) = (2l+1)h_l^{(1)}(z)/z\) implies

\[S = -3 \frac{h_1^{(1)}(\zeta)}{\zeta} P_1(\eta) + 5i \frac{h_2^{(1)}(\zeta)}{\zeta} P_2(\eta) + 7 \frac{h_3^{(1)}(\zeta)}{\zeta} P_3(\eta) - 9i \frac{h_4^{(1)}(\zeta)}{\zeta} P_4(\eta) + \ldots \]

\[- i^{-k+1}(2k+1) \frac{h_k^{(1)}(\zeta)}{\zeta} P_k(\eta) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \]

\[= - \sum_{n=1}^{k} i^{-n+1}(2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \]
which gives
\[
f_l(\eta, \zeta) = \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=1}^{k} i^{n+1}(2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \right) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) - i \frac{h_0^{(1)}(\zeta)}{\zeta} P_0(\eta) \right) \right\} i^l h_l^{(1)}(\zeta)
\]
or
\[
f_l(\eta, \zeta) = \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=0}^{k} i^{n+1}(2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \right) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) - i \frac{h_0^{(1)}(\zeta)}{\zeta} P_0(\eta) \right) \right\} i^l h_l^{(1)}(\zeta)
\]
This completes the lemma.

In conclusion, the functions \( h_l^\pm(z) \) defined in (4.2) can be expressed in the function \( h(\eta, \zeta) \) in (5.1). Specifically, we have
\[
h_l^\pm(z) = kah_l(z/a, \pm ka)
\]

6 Summary and explicit terms

This paper contains an evaluation of a non-trivial integral that occurs in the formulation of scattering by randomly distributed obstacles.

To summarize, the integral \( I_l(z) \) in (1.1) has been solved and the solution outside the interval \([-a, a]\) is a simple exponential function in \( kz \), while inside the interval \([-a, a]\), the solution can be found in a finite series of spherical waves. The finite sum of spherical waves depends on the two parameters \( kz \) and \( ka \), or, more precisely, the parameter \( ka \) and a polynomial of the order \( l \) in the parameter \( z/a \). Several equivalent solutions are presented in the paper, one of them is
\[
I_l(z) = \begin{cases} 
i^l e^{-ikz}, & z \leq -a \\
i^{-l}kah_0^{(1)}(ka)P_{l-2l/2}(z/a) + \sum_{n=0}^{\lfloor l/2 \rfloor -1} (-1)^n kah_{l-2n-1}^{(1)}(ka) (P_{l-2n}(z/a) - P_{l-2n-2}(z/a)), & z \in [-a, a] \\
i^{-l}e^{ikz}, & z \geq a \end{cases}
\]

The first integrals, \( l = 0, 1, 2, \ldots \), are of interest for low-frequency expansions. For \( l = 0 \) the integral is
\[
I_0(z) = \begin{cases} e^{-ikz}, & z \leq -a \\
e^{ika}, & z \in [-a, a] \\
e^{ikz}, & z \geq a \end{cases}
\]
and for \( l = 1 \) the result is
\[
I_1(z) = \begin{cases} 
  i e^{-ikz}, & z \leq -a \\
  -i e^{ika} \frac{z}{a}, & z \in [-a, a] \\
  -ie^{ikz}, & z \geq a
\end{cases}
\]

For \( l = 2 \) the result is
\[
I_2(z) = \begin{cases} 
  -e^{-ikz}, & z \leq -a \\
  e^{ika} (ka)^2 (3i + ka) - 3(i + ka)(ka)^2, & z \in [-a, a] \\
  -e^{ikz}, & z \geq a
\end{cases}
\]

and we notice that the integral contains a polynomial in \( z/a \) of order \( l \).

Moreover, the indefinite Fourier transform of \( I_1(z) \) has also been investigated. More precisely, the integral, see (4.1)
\[
\tilde{I}_1^\pm(z) = k \int_{z_0}^z I_l(t) e^{\pm ikt} \, dt, \quad z \geq z_0, \quad l = 0, 1, 2, \ldots
\]
is shown to have a solution expressed in spherical waves.

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**References**


