Spatially-Coupled Random Access on Graphs

Liva, Gianluigi; Paolini, Enrico; Lentmaier, Michael; Chiani, Marco

Published in:
[Host publication title missing]

DOI:
10.1109/ISIT.2012.6284235

2012

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Abstract—In this paper we investigate the effect of spatial coupling applied to the recently-proposed coded slotted ALOHA (CSA) random access protocol. Thanks to the bridge between the graphical model describing the iterative interference cancelation process of CSA over the random access frame and the erasure recovery process of low-density parity-check (LDPC) codes over the binary erasure channel (BEC), we propose an access protocol which is inspired by the convolutional LDPC code construction. The proposed protocol exploits the terminations of its graphical model to achieve the spatial coupling effect, attaining performance close to the theoretical limits of CSA. As for the convolutional LDPC code case, large iterative decoding thresholds are obtained by simply increasing the density of the graph. We show that the threshold saturation effect takes place by defining a suitable counterpart of the maximum-a-posteriori decoding threshold of spatially-coupled LDPC code ensembles. In the asymptotic setting, the proposed scheme allows sustaining a traffic close to $1$ [packets/slot].

I. INTRODUCTION

Since the introduction of the ALOHA protocol [1], several random access (RA) schemes have been introduced. Among them, some feedback-free RA protocols originally proposed in [2], [3] re-gained attention in the recent past [4], [5]. In [2], the capacity of the so-called collision channel without feedback (CCw/oFB) was analyzed, assuming slot-aligned but completely asynchronous users’ transmissions. Moreover, a simple approach to achieve error-free transmission (in noise-free setting) over the CCw/oFB was proposed. In the context of the CCw/oFB, the capacity is defined as maximum packet transmission rate per slot, which allows the receiver to recover the packets with an arbitrarily-small error probability (in noise-free conditions).

The approach of [2] consists of assigning different periodic protocol (access) sequences to the users. Each sequence defines in which slots each user is allowed to access the shared channel. Furthermore, the users encode their packets by means of erasure correcting codes. The user’s packet can be recovered whenever a sufficient number of codeword segments are received collision free. Hence, by selecting proper protocol sequences, it is possible to ensure that a sufficient number of segments per user are recovered, even if the beginning of the different protocol sequences is unsynchronized. In this way, a symmetric capacity\footnote{The symmetric capacity is given by the sum-rate capacity under the hypothesis that all users adopt the same transmission rate.} equal to $1/e$ [packets/slot] is achieved as $N \to \infty$, where $N$ is the number of users accessing the RA channel. The same capacity is achieved also in the unslotted case. Although simple, the approach of [2] poses some challenges, especially if a large (and varying) number of users has to be served [3], [4].

Recently, RA schemes profiting from successive interference cancelation (SIC) have been introduced and analyzed [6]–[9]. These schemes share the feature of canceling the interference caused by collided packets on the slots where they have been transmitted whenever a clean (uncollided) copy of them is detected. In [8], [9] it was shown that the SIC process can be well modeled by means of a bipartite graph. The analysis proposed in [8], [9] resembles density evolution analysis of low-density parity-check (LDPC) and doubly-generalized LDPC (D-GLDPC) codes over erasure channels [10]–[12]. By exploiting design techniques from the LDPC context, a remarkably-high capacity (e.g. up to $0.8$ [packets/slot]) can be achieved in practical implementations. The schemes considered in [6]–[8] assume a feedback from the receiver to achieve a zero packet loss rate.

A scheme based on the coded slotted ALOHA (CSA) of [9] has been analyzed in the context of the CCw/oFB in [13]. An upper bound on the maximum load $G$ sustainable at a scheme rate $R$, has been derived as the unique positive solution to

$$G = 1 - e^{-G/R}$$

in $[0, 1]$. Still in [13] it was shown how this bound can be tightly approached by a careful selection of the distribution of the codes to be used at users for encoding their packets.

In this paper, we propose another means for approaching the bound defined by (1), which is based on spatial coupling. Spatial coupling effects were initially devised in the context of density evolution analysis of convolutional LDPC codes over the binary erasure channel (BEC) [14]–[17] and the additive white Gaussian noise (AWGN) channel [18]. Subsequently, its application to other settings relying on sparse graph representations has been investigated (see e.g. [19]–[21]). By imposing some constraints on the CSA access scheme, we show how the threshold under the iterative (IT) SIC process saturates towards a suitably-defined equivalent of the maximum-a-posteriori (MAP) decoding threshold of LDPC ensembles.

II. CODED SLOTTED ALOHA: ERASURE DECODING MODEL

We recall next the basic model adopted for the description of CSA. We consider a slotted RA scheme where slots are grouped in medium access control (MAC) frames, all with

Gianluigi Liva is with Institute of Communication and Navigation of the Deutsches Zentrum für Luft- und Raumfahrt (DLR), 82234 Wessling, Germany (e-mail: Gianluigi.Liva@dlr.de).

Enrico Paolini and Marco Chiani are with CNIT, DEI, University of Bologna, 47521 Cesena (FC), Italy (e-mail: {e.paolini, marco.chiani}@unibo.it).

Michael Lentmaier is with the Vodafone Chair Mobile Communications Systems, Dresden University of Technology (TU Dresden), D-01062 Dresden, Germany (e-mail: Michael.Lentmaier@ifn.et.tu-dresden.de).

Supported in part by the EC under FP7 grant agreement n. 288502.
the same length (in slots). Each user is frame- and slot-
synchronous, and attempts at most one burst (i.e., packet) 
transmission per MAC frame. Each burst has a time duration 
$T_{\text{slot}}$, whereas the MAC frame is of time duration $T_{\text{frame}}$. 
Neglecting guard times, the MAC frame is composed of 
$M = T_{\text{frame}}/T_{\text{slot}}$ slots. We consider a population of $N$ 
users, with $N \gg M$. Users are characterized by a sporadic 
activity, i.e., at the beginning of a MAC frame each user 
generates a burst to be transmitted within the MAC frame with 
probability $\epsilon \ll 1$, where $\epsilon$ is called activation probability. 
Users attempting the transmission within a MAC frame are 
referred to as active users. On the contrary, users that are 
idle during a MAC frame are referred to as inactive users. 
We denote the population size normalized to the frame size by $\alpha = N/M$. The number of active users is modeled by 
the random variable $N_a$, which is binomially-distributed with 
mean value $\mathbb{E}[N_a] = N\epsilon$. We say that the average offered 
channel traffic (representing the average number of bursts 
transmissions per slot) is 

$$G = \mathbb{E}[N_a]/M = \epsilon N/M = \epsilon \alpha.$$ 

We consider a CSA scheme based on $(d, 1)$ repetition codes, 
which is equivalent to a $d$-regular contention resolution diversity 
slotted Aloha (CRDSA) scheme [6]. More specifically, at 
the beginning of a MAC frame, each user selects $d$ slots with 
a uniform probability out of the $M$ frame slots. If the user is 
active, it transmits $d$ copies of its burst in the $d$ selected slots. 
We define $R = 1/d$ as the rate of the scheme. In each burst 
replica, a pointer to the position of the other copies is included,

e.g., in a dedicated header field. Whenever a clean burst (i.e., 
a burst which did not collide) is successfully decoded, the 
pointer is used to determine the slots where its copies have 
been transmitted. Supposing that a another replica of this burst 
has collided, it is possible to subtract, from the signal received 
in the corresponding slot, the interference contribution of the 
twin burst. This may allow the decoding of another burst 
transmitted in the same slot. The SIC proceeds iteratively, i.e., 
cleaned bursts may allow solving other collisions. An example 
of a MAC frame with $M = 4$ slots and $N_a = 3$ active users 
is depicted in Fig. 1, where the repetition rate is $d = 2$.

Considering a MAC frame composed of $M$ slots and a 
population of $N = \alpha M$ users, the frame status can be 
described by a bipartite graph, $\mathcal{G} = (B, S, E)$, consisting of a 
set $B$ of $N$ burst nodes (one for each user), a set $S$ of $M$ sum 
nodes (one for each slot in the frame), and a set $E$ of edges. An 
edge connects a burst node $b \in B$ to a sum node $s \in S$ if and only if 
the $j$-th slot has been selected by the $i$-th user at the beginning of the MAC frame. The graph obtained 
by expurgating from $\mathcal{G}$ the BNs associated with inactive users 
and their adjacent edges is called the residual graph and is 
denoted by $\mathcal{G}_a = (B_a, S, E_a)$. Here, $B_a \subseteq B$ is the subset of 
BNs associated with the active users, and $E_a \subseteq E$ is the subset of 
the edges associated with the transmitted burst copies. An 
example of the residual graph representing the MAC frame of 
Fig. 1 is given in Fig. 2.

The SIC process can be represented through a message-
passing along the edges of the graph. As in [6], [8], we make 
use of two assumptions which allows simplifying the SIC 
process analysis in the graphical model. First, we assume that 
perfect SIC is performed. Second, we claim that, whenever a 
clean (collision-free) burst is present in a slot, decoding suc-
cedes with a probability that is essentially 1. It has been shown 
in [6], [8] that these assumptions are accurate enough to model 
the SIC process down to low signal-to-noise ratios (SNRs) 
with moderate-complexity signal processing algorithms.

Thanks to this simplification, the SIC procedure is equivalent 

to iterative decoding of an LDPC code with $N$ variable 
nodes and $M$ check nodes over a BEC with erasure probability 
$\epsilon$ (coinciding with the activation probability). All variable 
nodes have degree $d$, while the check node degrees follow a 
Poisson distribution [8] with average degree $dN/M = \alpha \epsilon$. 
The nominal code rate is thus $R_0 = 1 - M/N = 1 - 1/\alpha$.

For large frames ($M \rightarrow \infty$) and for a given normalized 
population size $\alpha$, CSA shows a threshold behavior. For an 
activation probability $\epsilon$ lower than a threshold value $\epsilon_{\text{block}}^2$, 
vanishing burst error probability can be achieved by iterating 
SIC. The threshold $\epsilon_{\text{block}}$ can be analyzed via density evolution 
over the residual graph $\mathcal{G}_a$ according to the recursions 

$$q_\ell = p_{\ell-1} / \rho_{\ell-1}$$

$$p_{\ell} = \sum_h \rho_h \left( 1 - (1 - q_\ell)^h - 1 \right) = 1 - \rho \left( 1 - q_\ell \right),$$

where $\rho = \rho_{\text{block}}^2$.

The subscript “block” is here used to emphasize the block-structure of the MAC frame, in contrast with the spatially-coupled structure introduced in Section III.
where $\hat{p}_h$ is the fraction of edges in $\mathcal{G}_h$ connected to SNs with degree $h$ in the residual code graph, and $\hat{p}(x) = \sum_h \hat{p}_h x^h$. In (2) and (3), $q_e$ and $p_t$ denote the probabilities that an edge in the residual graph carries an erasure outgoing from a BN and from a SN, respectively, at the $\ell$-th iteration. Since the number of collisions in a slot follows a Poisson distribution,

$$\hat{p}(x) = e^{-\hat{\rho}(1-x)}.$$  \hspace{1cm} (4)

Thus, the threshold $G_{\text{block}}$ is given by the supremum of the set of $\epsilon > 0$ such that

$$q > (1 - e^{-q\hat{\rho}d})^{\delta-1} \quad \forall q \in (0, 1].$$  \hspace{1cm} (5)

The threshold can be expressed equivalently in terms of offered traffic. By recalling that $G = \epsilon \alpha$, the threshold $G_{\text{block}}^{\text{IT}}$ is given by the supremum of the set of $G > 0$ such that

$$q > (1 - e^{-qGd})^{\delta-1} \quad \forall q \in (0, 1],$$  \hspace{1cm} (6)

and we have $G_{\text{block}}^{\text{IT}} = G_{\text{block}} \alpha$.

III. SPATIALLY-COUPLED CSA: ACCESS MODEL AND DENSITY EVOLUTION

In this section, we modify the access rules of CSA to implement a convolutional-oriented structure that enables the exploitation of the spatial coupling effect.

A. Access Model

The modified access rules are summarized next (see also Fig. 3). A super-frame is divided into $M_f = l + d - 1$ frames of $M$ slots each. The slots belonging to the same frame constitute a slot type set. A user becoming active at the beginning of a frame (with probability $\epsilon$) transmits a burst in a slot picked uniformly at random within that frame. Furthermore, a copy of the burst is sent in each of the following $d - 1$ frames in a slot picked with uniform probability in each frame. The set of users becoming active at the beginning of the $i$-th frame is referred to as the type-$i$ user set. Similarly, the slots belonging to the $j$-th frame are referred to as type-$j$ slots. The expected size of a user set is $\mathbb{E}[N_u] = \epsilon N$. Thus, as before we can define the offered traffic $G$ as $G = \mathbb{E}[N_u]/M = \epsilon N/M$.

After transmission of the $l$-th frame, transmissions from new users are forbidden, and the following $d - 1$ frames are filled just with the copies of the bursts whose transmissions have been initiated during the past $d - 1$ frames. Once all the burst copies have been transmitted, a new transmission cycle begins, i.e., a new super-frame is initialized.

A (residual) bipartite graph description of the recovery process is obtained as follows. We associate a BN to each user. Similarly, we associate a SN to each slot. The BNs corresponding to users of type $i$ are clustered in type-$i$ BN groups, whereas the SNs related to slots of type $i$ are clustered in type-$i$ SN groups. The number of BN types connected to a SN type-$j$ group is denoted by $\delta_j$ (degree of the type-$j$ SN group). Note that $\delta_j \in \{1, \ldots, d\}$. The type-$i$ BN group is said to be neighbor of a type-$j$ SN group (and vice versa) when the nodes belonging to the type-$i$ BN group are connected to some nodes in the type-$j$ SN group. The indexes of the

![Fig. 3. Example of a convolutional super-frame structure with 3 users per user type and $M = 4$ slots per frame.](image)

The modified access rules are summarized next (see also Fig. 3). A super-frame is divided into $M_f = l + d - 1$ frames of $M$ slots each. The slots belonging to the same frame constitute a slot type set. A user becoming active at the beginning of a frame (with probability $\epsilon$) transmits a burst in a slot picked uniformly at random within that frame. Furthermore, a copy of the burst is sent in each of the following $d - 1$ frames in a slot picked with uniform probability in each frame. The set of users becoming active at the beginning of the $i$-th frame is referred to as the type-$i$ user set. Similarly, the slots belonging to the $j$-th frame are referred to as type-$j$ slots. The expected size of a user set is $\mathbb{E}[N_u] = \epsilon N$. Thus, as before we can define the offered traffic $G$ as $G = \mathbb{E}[N_u]/M = \epsilon N/M$.

After transmission of the $l$-th frame, transmissions from new users are forbidden, and the following $d - 1$ frames are filled just with the copies of the bursts whose transmissions have been initiated during the past $d - 1$ frames. Once all the burst copies have been transmitted, a new transmission cycle begins, i.e., a new super-frame is initialized.

A (residual) bipartite graph description of the recovery process is obtained as follows. We associate a BN to each user. Similarly, we associate a SN to each slot. The BNs corresponding to users of type $i$ are clustered in type-$i$ BN groups, whereas the SNs related to slots of type $i$ are clustered in type-$i$ SN groups. The number of BN types connected to a SN type-$j$ group is denoted by $\delta_j$ (degree of the type-$j$ SN group). Note that $\delta_j \in \{1, \ldots, d\}$. The type-$i$ BN group is said to be neighbor of a type-$j$ SN group (and vice versa) when the nodes belonging to the type-$i$ BN group are connected to some nodes in the type-$j$ SN group. The indexes of the

groups that are neighbors of the type-$j$ SN group form the set $\mathcal{N}_j^B$, while the indexes of the groups that are neighbors of the type-$i$ BN group form the set $\mathcal{N}_i^S$. Note that the period in which new user transmissions are blocked is equivalent to the termination in the context of convolutional LDPC codes. An example of a super-frame structure is displayed in Fig. 3. The bursts transmitted into termination frames experience a lower collision probability than the other bursts, thus boot-strapping the iterative decoding process through the coupled structure.

B. Density Evolution

Let $p_j$ be the probability that an edge incident on the type-$j$ SN group carries an erasure message towards the BNs, after SN processing at the generic SIC iteration. Analogously, let $q_j$ be the probability that an edge incident on the type-$j$ SN group carries an erasure message towards the type-$j$ SNs, after BN processing at the generic SIC iteration. Moreover, let $q_{i\rightarrow j}$ be the probability that an edge emanating from the type-$i$ BN group carries an erasure message towards the type-$j$ SN group (with $j \in \mathcal{N}_i^S$), after BN processing at the generic SIC iteration. The physical load (i.e., the load including burst copies) for the $i$-th sub-frame is given by $G^{(i)} = G \cdot \delta_i$.

Next, we define SN degree distributions from an edge perspective as

$$\rho^{(j)}(x) = \sum_{t=0}^{\infty} \rho_t^{(j)} x^{t-1} = \exp \left( -G \delta_j(1-x) \right)$$

where $\rho_t^{(j)}$ is the fraction of the edges emanating from type-$j$ SNs and incident on type-$j$ SNs with degree $t$. Density evolution equations can be now derived as follows, where $l$ is the iteration index. For the type-$j$ SN group we have

$$p_{j,l} = 1 - \rho^{(j)}(1 - \delta_j)$$

where

$$q_{j,l} = \frac{1}{\delta_j} \sum_{w \in \mathcal{N}_j^S} q_{w \rightarrow j,l}.$$
A matrix representation of the graph via an
schemes, and to investigate threshold saturation effects for
derive an upper bound on the achievable threshold for CSA
mitted bursts and the slot observations can be simplified by t o
A. Genie-Aided MAP Decoding
decoder of an LDPC code over the BEC, and we refer to it
the convolutional scheme. This algorithm mimics the MAP
close to 1 [packet
be tightly approached by spatially-coupled CSA based on non -
The large SIC IT thresholds attained by the convolutional CSA
over the BEC for the ensemble \( \mathcal{C}_{d,M,N}, N \to \infty \), is calculated as the maximum value of the channel erasure probability \( \epsilon \) (the analogous of the activation probability, in the CSA context) for which the erasure probabilities \( q_i, p_i \) (where \( i \) is the iteration index) converge to an arbitrarily-low positive value, for \( i \to \infty \), according to
\[
p_i = \epsilon q_i^{d-1}, \\
q_i = 1 - \rho(1 - p_i) = 1 - \exp(-d_c p_i).
\]
The average extrinsic erasure probability \( p_e(\epsilon) \) under IT decoding is obtained finally as
\[
p_e^{\text{IT}}(\epsilon) = \lim_{i \to \infty} q_i^d.
\]
B. CSA Analysis under GA-MAP Decoding

We establish next a bridge towards the MAP decoding
threshold of LDPC codes under MAP decoding in order to
derive the threshold of a \( d \)-regular CSA scheme under GA-MAP decoding, \( G_{\text{MAP}} \). We define \( \mathcal{C}_{d,M,N} \) to be the ensemble of all length-\( N \) binary parity-check matrix \( H \), having constant column weight \( d \) and where the \( d \) 1s in each column are placed in random positions, according to a uniform distribution. Recall that, for the codes in this ensemble, the nominal rate is given by \( R_0 = 1 - M/N \). From a bipartite graph perspective, the graph of a code in \( \mathcal{C}_{d,M,N} \) possesses a constant variable node degree, \( d_v = d \) whereas, as \( N \) and \( M = (1 - R_0)N \) tend to infinity, the check node degree distribution follows a Poisson distribution with mean value \( d_c = dN/M \). The edge-oriented check node degree distribution is thus given by \( \rho(x) = \exp(-d_c (1 - x)) \) [8].

Recall that the ensemble under consideration can be placed in analogy to the scheme introduced in Section II where \( N \) is the user population size, \( M \) is the number of slots per frame and \( d \) is the repetition rate for the bursts. The IT decoding threshold \( IT_{\text{block}} \) over the BEC for the ensemble \( \mathcal{C}_{d,M,N}, N \to \infty \), is calculated as the area below the curve, which is given by the function \( \rho(x) \) for \( x = 0 \) to \( x = 1 \).
noting that for any \( \epsilon, p_e^{\text{MAP}}(\epsilon) \leq p_e^{\text{IT}}(\epsilon) \), an upper bound [23] on \( \epsilon^{\text{MAP}} \) is given by the value \( \epsilon^{\text{IT}} \), such that

\[
\int_0^1 p_e^{\text{IT}}(\epsilon) d\epsilon = R_0.
\]

This allows us also to get an upper bound on the decoding threshold for a \( d \)-regular block CSA scheme, under GA-MAP decoding. Letting \( \alpha = N/M = 1/(1 - R_0) \), the GA-MAP threshold of CSA can be upper bounded as

\[
\epsilon^{\text{MAP}} = \alpha \epsilon^{\text{MAP}}.
\]

C. Threshold Saturation

Table I illustrates the threshold achievable by conventional CSA schemes employing a regular distribution at the BNs based on \((d,1)\) repetition codes. For the spatially-coupled scheme, a super-frame composed by \( M_f = l + d - 1 \) frames has been considered, with \( l = 200 \). Moreover, the normalized user population size is \( \alpha = 100 \), i.e. the number of users is 100 times larger than the number of slots per frame. We additionally provide the upper bounds on the threshold achievable by the conventional CSA scheme under the GA-MAP recovery process. The derivation of the MAP thresholds serves to illustrate how, also in this context, the imposition of a convolutional-like structure to the access scheme allows achieving the threshold saturation effect as numerically shown in Table I. The upper bound on the achievable threshold \( G^* \) according to (1), given by the solution of \( G = 1 - \exp(-G/R) \), is provided too. Accordingly, we evaluated the normalized efficiency of the proposed scheme as

\[
\eta = \epsilon^{\text{MAP}} / G^*.
\]

As already observed in the LDPC context, larger degrees allow to approach the bound more tightly.

### Table I

<table>
<thead>
<tr>
<th>( d )</th>
<th>( G^{\text{IT}}_{\text{block}} )</th>
<th>( \epsilon^{\text{IT}}_{\text{conv}} )</th>
<th>( \epsilon^{\text{MAP}}_{\text{block}} )</th>
<th>( G^* )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.960</td>
<td>0.3726</td>
</tr>
<tr>
<td>3</td>
<td>0.8184</td>
<td>0.9179</td>
<td>0.9179</td>
<td>0.9405</td>
<td>0.9760</td>
</tr>
<tr>
<td>4</td>
<td>0.7722</td>
<td>0.9767</td>
<td>0.9767</td>
<td>0.9802</td>
<td>0.9964</td>
</tr>
<tr>
<td>5</td>
<td>0.7017</td>
<td>0.9924</td>
<td>0.9924</td>
<td>0.9931</td>
<td>0.9993</td>
</tr>
<tr>
<td>6</td>
<td>0.6370</td>
<td>0.9973</td>
<td>0.9973</td>
<td>0.9975</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

V. CONCLUSION

In this paper we introduced a spatially-coupled RA scheme for the CC/w/oFB which attains capacities close to 1 [packet/slot] in the asymptotic (i.e., for large frames) setting. A bridge between the graphical model describing the iterative interference cancelation process of the proposed RA over the random access frame and the erasure recovery process of low-density parity-check codes over the binary erasure channel has been set, which allows computing an upper bound on the capacity achievable by an enhanced (genie-aided) decoder. The saturation of the SIC IT capacity of the proposed scheme towards the threshold under genie-aided decoding has been numerically demonstrated.

### REFERENCES