Improving Code Diversity on Block-Fading Channels by Spatial Coupling

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Abstract—Spatially coupled low-density parity-check (SC-LDPC) codes are considered for transmission over the block-fading channel. The diversity order of the SC-LDPC codes is studied using density evolution and simulation results. We demonstrate that the diversity order of the code can be increased, without lowering the code rate, by simply increasing the coupling parameter (memory) of a SC-LDPC code. For a (3,6)-regular SC-LDPC code with rate $R = 1/2$ and memory $m_{sc} = 4$, a remarkable diversity of $d = 10$ is achieved without the need for any specific code structure. The memory of the SC-LDPC codes makes them robust against a non-stationary mobile-radio environment. The decoding of SC-LDPC codes using a latency constrained sliding window decoder is also considered.

I. INTRODUCTION

The mobile-radio channel can be modelled as a slow, flat fading together with additive noise. In many cases, the channel coherence time is much longer than one symbol duration. Thus several symbols are affected by the same fading coefficient. An example of such a channel model is the block-fading channel introduced in [1]. In block-fading channel, coded information is transmitted over a finite number of fading blocks to provide diversity. The diversity order of the code is an important parameter that gives the slope of the word error rate (WER) of the decoder. Codes achieving diversity equal to the number of fading blocks in a codeword are said to be full diversity codes. In [2], a family of LDPC block codes, called root-LDPC codes, are proposed that provide full diversity over a block-fading channel. The root-LDPC codes have a special check node structure called rootcheck. Full diversity is provided to the systematic information bits by connecting only one information bit to every rootcheck.

In this paper, we consider spatially coupled low-density parity-check (SC-LDPC) codes for block-fading channels based on the following two observations,

- Convolutional codes, in general, are known to be suitable for transmission over block-fading channels and the diversity can be increased by increasing the constraint length of the code [3].
- Root-LDPC codes provide full diversity for the information bits. However, designing root-LDPC codes with diversity order greater than 2 requires codes with rate less than 1/2. The special structure of the codes makes it a complicated task to generate good root-LDPC codes with high diversity (and thus low rate).

Since SC-LDPC codes have a convolutional structure, they are expected to have a good performance over block-fading channels. In this paper, we present density evolution outage probabilities [2] for random LDPC block and SC-LDPC codes in Section III and IV, respectively. The results suggest that increasing the coupling parameter (constraint length) of the SC-LDPC code improves the diversity order of the code. A similar observation is made in [4] for the block erasure channel, which is a special case of block-fading channel.

Furthermore, we show that a higher diversity order can be achieved without decreasing the code rate $R$ and that SC-LDPC codes are robust against the variations in the channel, e.g. a mobile-radio channel is not stationary over time and it fluctuates between the extremes of Rayleigh and AWGN channel. We support the density evolution results with simulation results for finite length codes. In Section V, a latency constrained sliding window decoder [5] for the SC-LDPC code is considered. It has been shown in [6] for an AWGN channel, that even for a small structural delay (< 500 bits), the windowed decoding of SC-LDPC codes outperform both conventional convolutional codes with Viterbi decoding and LDPC block codes. We consider (3,6)-regular codes as an example, which already exhibit remarkable diversity improvement. A general analytical bound on achievable diversity is hard to obtain but our experimental results indicate that better diversity can be achieved by further increasing the node degrees.

II. NON-ERGODIC BLOCK-FADING CHANNEL

We consider transmission of a codeword of length $N$. The slowly varying nature of the channel allows us to divide the codeword into $F$ subblocks, of length $N_f = N/F$ each. These $F$ subblocks are affected by different independent fading realizations $\alpha_j, j = 1, \ldots, F$. The received symbols, $y_i, i = 1, \ldots, N$, have the following form,

$$y_i = \alpha_j x_i + n_i, \quad (1)$$
where \( j = 1 + \lfloor (i-1)/N_f \rfloor \) and \( \lfloor \cdot \rfloor \) represents the floor operator. The input symbols \( x_i \) are chosen from BPSK alphabet and \( n_i \) are Gaussian random variables with zero mean and variance \( \sigma_n^2 \). The symbols are normalized to \( x_i = \pm 1 \) and the fading coefficients \( \alpha_j \) are Rayleigh distributed with \( \mathbb{E}[\alpha_j^2] = 1 \). Hence, the signal-to-noise ratio (SNR= \( \gamma \)) of the received symbols is characterized only by the variance of the Gaussian noise. Figure 1 gives an example of a codeword transmitted over two fading channels (\( F = 2 \)). The fading values \( \alpha_1 \) and \( \alpha_2 \) are constant throughout the first and second half of the codeword, respectively. We further assume perfect channel information only at the receiver side\(^1\). The log-likelihood ratio (LLR) of the received symbols \( y_i \) is given as,

\[
L_i = \frac{2\alpha_j}{\sigma_n^2} y_i .
\]  

The capacity of a non-ergodic channel depends on the channel realization (here fading coefficients \( \alpha_j \)) and hence is not information stable [7]. Therefore, the Shannon capacity of a block-fading channel is zero. In order to characterize such a channel, the outage probability \( P_{out} \) serves as a lower bound on the word error probability for any coding scheme. The outage probability is defined as follows [1],

\[
P_{out} = \mathbb{P}(I(\mathbf{x};\mathbf{y}|\alpha) < R)
\]

where \( I(\cdot) \) denotes the instantaneous mutual information between the input and the output of the channel, \( R \) is the transmission rate and \( \alpha = \{\alpha_1,\ldots,\alpha_F\} \). Hence, an outage occurs when the fading values in \( \alpha \) are such that the mutual information between the input and the output of the channel is below the code rate. Assuming the input symbols \( x_i \) to be binary (BPSK), the outage probability is given as, [8],

\[
P_{out} = \mathbb{P} \left( \sum_{j=1}^{F} \mathbb{E}[\log_2(1 + e^{-4R\alpha_j^2 X})] > F(1 - R) \right)
\]

with \( X \sim N(\gamma, \gamma) \). The \( P_{out} \) in (4) is an outage boundary for a random code and can be approximated to \( 2/\gamma^2 \) for \( R = 1/2 \) code and \( F = 2 \) under high SNR(\( \gamma \)) [8]. However, in general \( P_{out} \) has no close form expression.

**A. Coding for Block-Fading Channels**

A code design for a block-fading channel must exploit the distinct characteristics of the channel. An important feature is that each subblock is faded by an independent fading value, providing some diversity. The \( F \) subblocks in a codeword are coded together to reduce the probability of all the code symbols being faded simultaneously. At high SNR, the diversity order of the code defines the slope of the error rate curve on a log-log scale and is given as,

\[
d = -\lim_{\gamma \to \infty} \frac{\log(P_{we})}{\log(\gamma)}
\]

where \( P_{we} \) is the word error probability. A code with rate \( R = 1/F \) is said to have full diversity if \( d = F \). Next, we consider an LDPC block code and a spatially coupled LDPC code for block-fading channel.

**III. LDPC Codes for Block-Fading Channels**

A \((J, K)\)-regular LDPC code is characterized by a sparse parity-check matrix \( H \) containing exactly \( J \) and \( K \) ones in each column and row, respectively. Here, we consider protograph based LDPC codes described by a bi-adjacency matrix \( B \), called base matrix. A protograph is a small bipartite graph consisting of \( n_c \) check and \( n_v \) variable nodes. The parity-check matrix \( H \) of an LDPC code can be obtained by applying a lifting procedure that replaces each \( 1 \) in \( B \) by an \( Z \times Z \) permutation matrix and each \( 0 \) by an \( Z \times Z \) all-zero matrix. Integer entries larger than \( 1 \) represent multiple edges between a pair of nodes and are replaced by a sum of permutation matrices. The resultant parity-check matrix \( H \) defines a codeword \( \nu = \{v_1,\ldots,v_N\} \) of length \( N = Zn_v \).

**A. Density Evolution Outage**

We use density evolution to analyze the exact performance of a random LDPC block code for a block-fading channel. Density evolution tracks the probability density function of the messages exchanged between the check and variable nodes in the bipartite graph. The worst channel parameter for which the bit error probability converges to zero is called the threshold of an ensemble. Here, as discussed before, the threshold of an ensemble for the block-fading channel depends on the channel realization and hence does not exist.

Considering (3), an outage occurs when instantaneous input-output mutual information is less than the transmission rate \( R \). In terms of density evolution, we define density evolution outage (DEO) as an event where the bit error probability does not converge to zero for a fixed value of SNR after finite or infinite number of decoding iterations are performed [2]. The probability of density evolution outage, \( P_{DEO} \), for a fixed value of SNR can then be calculated using a Monte Carlo method considering significant number of fading coefficients. The lower bound on the word error probability \( P_{we} \) is given as [2],

\[
P_{we} \geq P_{DEO}.
\]

Consider, without loss of generality, transmission of all-zero codeword and perfect channel information at the receiver side. The mean and variance of the received LLR from (2) can then be calculated as \( 2\alpha_j^2/\sigma_n^2 \) and \( 4\alpha_j^2/\sigma_n^2 \), respectively. Hence, the block-fading channel can be modeled as an AWGN channel with gain \( \alpha_j \) for each subblock of length \( N_f \) symbols and the initial distribution of the received symbols is generated using

Fig. 1. Illustration of block-fading channel for a codeword of length \( N \) and \( F = 2 \).
the mean and variance of the LLR value in (2). The fading coefficient $\alpha_i$ can be interpreted as a known gain.

Figure 2 shows the outage probability bound for random block codes and $P_{DEO}$ for a random (3,6)-regular LDPC codes calculated using (4) and (6), respectively. It can be observed that the LDPC block code does not achieve the outage bound and its diversity order is $d \approx 1.3$. In order to achieve full diversity $d = F = 2$, which is the maximum achievable diversity order for an $R = 1/F$ code, root-LDPC codes are introduced in [2]. The simulation result for a randomly generated (3,6)-regular root-LDPC block code to calculate the density evolution for a SC-LDPC is far too complex, we use a protograph based on a randomly generated (3,6)-regular root-LDPC code in Fig. 2 shows that the WER closely matches with the outage bound, corresponding to a diversity $d = 2$.

IV. SC-LDPC CODES FOR BLOCK-FADING CHANNELS

Now consider the transmission of a sequence of codewords $v_t, t = 1, \ldots, L$ each of length $N$, using a protograph based LDPC code. Instead of encoding the sequence of codewords independently, the blocks $v_t$ are coupled by the encoder over various other time instants [9]. The maximal distance between a pair of coupled blocks defines the memory $m_{cc}$ of the convolutional code. The coupling of consecutive blocks can be achieved by an edge spreading procedure [10] that distributes the edges from variable nodes at time $t$ among equivalent check nodes at times $t+i, i = 0, \ldots, m_{cc}$. This procedure is illustrated in Fig. 3 for a (3,6)-regular protograph with base matrix $B = [3,3]$. In order to maintain the degree distribution and structure of the original ensemble, a valid edge spreading should satisfy the condition $\sum_{i=0}^{m_{cc}} B_i = B$. The resulting ensemble can be described by means of a terminated convolutional protograph with base matrix

$$B_{[1,L]} = \begin{bmatrix}
    B_0 & \cdots & B_0 \\
    B_{m_{cc}} & \cdots & B_{m_{cc}} \\
    \vdots & \ddots & \vdots \\
    B_{(L+m_{cc})m_{cc} \times Lm_v}
\end{bmatrix}. \quad (7)$$

The corresponding sequence of coupled code blocks forms a codeword $v_{[1,L]} = [v_1, v_2, \ldots, v_t, \ldots, v_L]$ of a terminated SC-LDPC code. Note that the $m_{cc}N_f$ additional check nodes result in a rate loss due to termination. The block code ensemble with disconnected protographs corresponds to the special case $m_{cc} = 0$ with $B_0 = B$.

A. Density Evolution Outage for SC-LDPC Codes

Similar to LDPC block codes in Section III, DEO can be calculated for a SC-LDPC code represented by a coupled bipartite graph in Fig. 3. An illustration of a terminated SC-LDPC codeword $v_{[1,L]}$ with termination length $L$. The length of each coupled codeword $v_t$ is $N$.

SC-LDPC code. Note that the $m_{cc}N_f$ additional check nodes result in a rate loss due to termination. The block code ensemble with disconnected protographs corresponds to the special case $m_{cc} = 0$ with $B_0 = B$.

Since a Monte Carlo method to calculate $P_{DEO}$ with exact density evolution for a SC-LDPC is far too complex, we use a reciprocal channel approximation (RCA) technique [11] to calculate the $P_{DEO}$ for a fixed SNR value. In case of the BEC, density evolution can be represented by a one-dimensional parameter, i.e., erasure probability. RCA uses a one-dimensional representation per variable node for the block-fading channel. The one-dimensional parameter within the RCA method is the mean of the received LLR symbol given by (2) and depends on the particular fading realization.

A check node at time $t$ in a SC-LDPC code is connected to $m_{cc}+1$ codewords $v_{t-i}, i = 0, \ldots, m_{cc}$ as shown in Fig. 3. Considering the block-fading model in Fig. 4, a codeword $v_t$ has $F$ independent fading values. Hence, each check node in the coupled graph is connected to at most $F(m_{cc}+1)$ independent fading coefficients. Therefore, for a fixed channel parameter $F$, the diversity order can be increased by increasing $m_{cc}$ of the coupled code, while keeping $R$ and $F$ same.

The component matrices for a (3,6)-regular SC-LDPC code with an increasing memory $m_{cc} = 1, \ldots, 4$ are considered in Table I. Figure 5 shows the corresponding DEO probabilities determined using (6) and RCA approximation. The diversity
in order to achieve the diversity order of 

sufficient to achieve the diversity order of 
generated (3,6)-regular SC-LDPC code with 
but only with a rate 

Fig. 2, we can conclude that a random LDPC block code with 
N 

optimal maximum likelihood decoding, a codeword of length 
(4). The results suggest that in order to achieve the same 
F 

diversity can be achieved with a block code in case of 
the code is 

increasing node degree and memory would result in increase 
the diversity order. This is due to the fact that the maximum 
number of codewords connected to a check node is limited by 
the memory and the node degree. Hence, only simultaneously 
increasing node degree and memory would result in increase 
in the diversity order of the ensemble.

Considering EnsA2 as an example, the diversity order of 
the code is d = 6 with R = 1/2. As shown in Fig. 6, the 
same diversity can be achieved with a block code in case of 
F = 10 according to the outage bound, as calculated using 
(4). The results suggest that in order to achieve the same 
performance with a random block code of R = 1/2 and 
optimal maximum likelihood decoding, a codeword of length 
N = 10N_f (F = 10) must be considered. Note that from 
Fig. 2, we can conclude that a random LDPC block code with 
N = 10N_f will not reach this outage bound. However, a full-
diversity root-LDPC code with d = 6 is possible to design 
but only with a rate R = 1/d = 1/6. In contrast to this, at 
the expense of slight rate loss due to termination, a randomly 
generated (3,6)-regular SC-LDPC code with m_{cc} = 2 is 
sufficient to achieve the diversity order of d = 6. Likewise, 
in order to achieve the diversity order of d = 10, similar to 
EnsA3 and EnsA4 (see Table I), a rate R = 1/2 random block 
code with F = 16 is required (curve not shown here).

Figure 6 also shows the DEO probability for F = 1, where 
N = N_f. The diversity order of EnsA2 reduces to 3. Whereas, 
a block code with R = 1/2 or a root-LDPC code designed 
for F = 2 would have a maximum diversity of d = 1^2. This 
suggests that the SC-LDPC codes are more robust against 
the variation in the channel parameter F compared to root-
LDPC codes, i.e., designing a code for a specific value of F 
is not required and the diversity order strongly depends on 
the memory of the code. We observed that as long as the 
constraint length of the code N(m_{cc} + 1) contains more than 
one subblock of length N_f, the SC-LDPC code can provide 
a diversity order greater than 1. However, so far we have not 
been able to give any bound on the diversity of a SC-LDPC 
code with respect to the coupling parameter and more analysis 
is required.

**B. Simulation Results**

We demonstrate the results of WER for finite length codes 
generated randomly while avoiding the cycles of length 4. The 
block length of each individual coupled code v_i is N = 200 
and F = 2. A maximum number of 50 iterations are performed 
and the iterations stop once the check nodes are fulfilled. 
Figure 7 shows the simulated WER for EnsA1, EnsA2 and 
EnsA4 together with the DEO probabilities from Fig. 5. We 
observe that there is no significant difference between the DEO 
probability and the simulated WER for a finite block length N. 
Note that, density evolution assumes an infinite block length.

### V. LATENCY CONSTRAINED DECODING OF SC-LDPC CODES

So far, we presented the results when standard belief prop-
gagation decoding is applied across the codeword v_{[1:L]}. This 
induces a large structural decoding delay of LN code bits. 
The structural delay is defined as the number of code bits, the 
decoder has to wait before starting the decoding process. In 
order to limit the decoding latency, we use a sliding windowed 
decoder of size W introduced in [5].

\[^2\text{using Singleton-like bound [12] on diversity order }d \leq 1 + \left\lfloor (1 - R) \right\rfloor\]
A window at time $t$ operates on $W$ received words, $y_1, y_{t+1}, \ldots, y_{t+W-1}$, corresponding to a section of $Wn_c$ rows and $Wn_v$ columns of the matrix in (7). The size of the window decoder is limited by at least $m_{cc} + 1$ codewords, which is the maximal distance between two coupled codewords. At window position $t$, only symbols in $y_t$ are decoded and hence termed as target symbols. After the received word $y_t$ is decoded or the maximum number of iterations are performed, the window slides $n_c$ rows down and $n_v$ columns right in $B_{[t,L]}$. By using a window decoder, the structural latency is reduced to $WN$, where in general $W < L$.

The window decoder of size $W$ is applied to the same codes used in Fig. 7. A maximum number of 50 iterations are performed at each window position. We compare the results for EnsA1 and EnsA2 for different window sizes. The edge spreading in EnsA1 containing multiple edges in $B_0$ is considered as this achieves the target BER with smaller $W$ [5] for an AWGN and BEC channel. Note that the structural latency is proportional to $W$ and hence a small value of $W$ is desirable for latency constrained applications.

Figure 8 shows the WER when a window decoder is used to decode a SC-LDPC code. The corresponding DEO probabilities from Fig. 5 are also plotted as reference. We demonstrate that SC-LDPC codes do not require channel specific design and the increase in diversity order depends on the coupling parameter $m_{cc}$ of the code. While root-LDPC codes have to be designed specifically for a given $F$, the SC-LDPC codes are more robust against variation in the fading channel. Furthermore, a latency constrained decoding of SC-LDPC codes using a window decoder gives no significant loss in word error performance compared to a block decoder.

**VI. CONCLUSION**

We present DEO probabilities and simulation results on finite length SC-LDPC codes for the block-fading channel. A simple $(3,6)$-regular SC-LDPC codes have been demonstrated to achieve a diversity order of $d = 10$. We demonstrate that SC-LDPC codes do not require channel specific design and the increase in diversity order depends on the coupling parameter $m_{cc}$ of the code. While root-LDPC codes have to be designed specifically for a given $F$, the SC-LDPC codes are more robust against variation in the fading channel. Furthermore, a latency constrained decoding of SC-LDPC codes using a window decoder gives no significant loss in word error performance compared to a block decoder.

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